ISSN:2473-6988

AIMS MATHEMATICS

VOLUME NO. 9
ISSUE NO. 1
JANUARY - APRIL 2024

ENRICHED PUBLICATIONS PVT. LTD

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MANISH ABHINAV PLAZA-II, ABOVE FEDERAL BANK,
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PHONE: - + (91)-(11)-47026006

ISSN:2473-6988

AIMS MATHEMATICS

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ISSN:2473-6988

ISSN:2473-6988

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ISSN:2473-6988

ISSN:2473-6988

ISSN:2473-6988

ISSN:2473-6988

ISSN:2473-6988

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(Volume No. 9, Issue No. 1, January - April 2024)

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Mittag-Leffler stabilization of anti-periodic solutions for fractionalorder neural networks with time-varying delays

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A B S T R A C T

Mittag-Leffler stabilization of anti-periodic solutions for fractional-order neural networks with time-varying delays are investigated in the article. We derive the relationship between the fractional-order integrals of the state function with and without delays through the division of time interval, using the properties of fractional calculus, and initial conditions. Moreover, by constructing the sequence solution of the system function which converges to a continuous function uniformly with the Arzela-Asoli theorem, a sufficient condition is obtained to ensure the existence of an anti-periodic solution and Mittag-Leffler stabilization of the system. In the final, we verify the correctness of the conclusion by numerical simulation.

Keywords: fractional-order; time-varying delays; anti-periodic solutions; Mittag-Leffler stabilization; neural networks
Mathematics Subject Classification: 92B20, 34K20

1. Introduction

The stabilization and existence of anti-periodic solutions have major significance in dynamic behavior on nonlinear differential equations, which plays a key role in various physical phenomena, such as anti-periodic characteristics in vibration equations and so on [1–5]. As a special case of periodic solutions, many scholars have studied the existence and stabilization of anti-periodic solutions of several kinds of neural networks in recent years. The authors [6] studied the existence and stabilization of anti-periodic solutions for BAM Cohen-Grossberg neural networks. In [7] authors investigated the existence and global exponential stabilization of anti-periodic solutions for quaternion numerical cellular neural networks with impulse effect. The existence and exponential stabilization of antiperiodic solutions for BAM neural networks is studied in [8,9]. The authors [10] studied the global exponential stabilization of anti-periodic solutions for Cohen-Grossberg neural networks. All studies in [6–10] are integer-order models, however, the research on fractional-order neural networks has attracted attention and obtained important research results in recent years.

The existence and stabilization of anti-periodic solutions are of great significance in the dynamic behavior of nonlinear differential equations, such as [1–5]. From previous data, there are only discussions on the asymptotic ω -periodic solution, almost periodic solutions and s -asymptotic ω periodic solutions for fractional-order neural networks (e.g., [11–15]), we haven't found the existence and stabilization of anti-periodic solutions yet. We focus on the problem of the existence of antiperiodic solutions and Mittag-Leffler stabilization for a class of fractional order neural networks in this paper, this is a new research topic, our characteristics mainly include three points:

1) Deriving the relationship between fractional-order integrals of state functions with and without time delay through the division of time interval and the properties of fractional-order calculus;

2) Constructing function sequence solution, and it uniformly converges to a continuous function with Arzela-Asoli theorem, then giving a sufficiency for the existence of anti-periodic solutions and Mittag-Leffler stabilization of the system, the results are new；

3) Verifying the correctness of the theorems by numerical simulation instances. It provides a new criterion for dynamic system research. We consider fractional-order neural networks with time-varying delays:

$$D_t^{\alpha} x_i(t) = -\beta_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t-\tau_{ij}(t))) + I_i(t), i = 1,2,\cdots,n. \quad (1)$$

Where t ≥ 0, Dtαis Riemann-Liouville derivative with α -order, 0 1 < α < ; (t) ix is the state of the Ith neuron at time t; β 0 > i ; ,ij ij a bare connection weights of neurons; (.) fj is an excitation function of the jth neuron; (t) iI tis an external input function of the Ith neuron at time t; π ij (t) is a signal transmission delay between theIth neuron and the jth neuron, and π ij (t)>0 .Given the initial conditions of the system (1):

$$x_i(s) = \varphi_i(s),\ D_t^{\alpha} x_i(s) = \psi_i(s),\ -\tau \le s \le 0,\ i = 1,2,\cdots,n. \quad (2)$$

Here $\tau = \sup_{1\le i,j\le n,\, t>0} \{\tau_{ij}(t)\}$, $\varphi_i(s), \psi_i(s)$ are bounded continuous functions.

The structure of this article is as follow. First a few preliminaries are given in Section 2. In Section 3, by the properties of fractional-order calculus, constructing function sequence solution, and the ArzelaAsoli theorem, a sufficient case is derived for the existence of anti-periodic solutions and Mittag Leffler stabilization of the system. An illustrative example to show the effectiveness of the proposed theory in Section 4.

2. Preliminaries

Definition 1. [16] Define the q order fractional-order integral of f (t)(Riemann-Liouville integral) as

$$D_t^{-q} f(t) = \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-r)^{q-1} f(r) \mathrm{d}r,$$

where $t \ge t_0 \ge 0$, q is a positive real number, $\Gamma(\cdot)$ is a Gamma function, and $\Gamma(r) = \int_0^{+\infty} t^{r-1} e^{-t} \mathrm{d}t, r > 0.$

Definition 2. [16] Define the q -order fractional-order derivative of *f(t)* (Riemann-Liourille derivative) as

$$D_t^q f(t) = \frac{1}{\Gamma(n-q)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_{t_0}^{t} \frac{f(s)}{(t-s)^{q-n+1}} \mathrm{ds},$$

where $t \ge t_0 \ge 0$, $n-1 \le q < n, n \in Z^+$, $\Gamma(\cdot)$ is a Gamma function.

Definition 3. [17] A Mittag-Leffler function with parameter q is defined

$$E_q(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(kq+1)},$$

where $R\mathrm{e}(q) > 0$ is the real part of q, z is plural, $\Gamma(\cdot)$ is a Gamma function.

Definition 4. Let $X^{\mathrm{T}}(t)$ and $\bar{X}^{\mathrm{T}}(t)$ are the solutions of $x_i(s) = \varphi_i(s), D_t^\alpha x_i(s) = \psi_i(s)$ and $\bar{x}_i(s) = \bar{\varphi}_i(s)$, $D_t^\alpha \bar{x}_i(s) = \bar{\psi}_i(s)$, $-\tau \le s \le 0$. If there exist $\rho_1 > 0$, $\rho_2 > 0$, $\bar{X}^{\mathrm{T}}(t)$ and $X^{\mathrm{T}}(t)$ satisfy

$$\left\| X(t) - \bar{X}(t) \right\| \le [M(\varphi - \bar{\varphi}) E_q(-\rho_1 t^{\rho_1})]^{\rho_2}, t \ge 0,$$

then the system (1) is Mittag-Leffler stabilization, where
$X(t) = (x_1(t), x_2(t), \cdots x_n(t))^{\mathrm{T}}$, $\bar{X}(t) = (\bar{x}_1(t), \bar{x}_2(t), \cdots \bar{x}_n(t))^{\mathrm{T}}$, $\varphi(t) = (\varphi_1(t), \varphi_2(t), \cdots \varphi_n(t))^{\mathrm{T}}$,
$\bar{\varphi}(t) = (\bar{\varphi}_1(t), \bar{\varphi}_2(t), \cdots \bar{\varphi}_n(t))^{\mathrm{T}}$, $M(\varphi - \bar{\varphi}) \ge 0, M(0) = 0.$ $E_q(\cdot)$ is a Mittag-Leffler function with a parameter q.

Lemma 1. [18] $x(t)$ is a continuously differentiable function on $[0, \delta](\delta > 0)$, then
$D_t^{-p} D_t^q x(t) = D_t^{-p+q} x(t)$, $0 < q < 1, n-1 \le p < n, n \in Z^+$.

Lemma 2. [19] $u(t)$ is a continuous function on $[0, +\infty)$, there exists $d_1 > 0$ and $d_2 > 0$, such that $u(t) \le -d_1 D_t^{-q} u(t) + d_2$, $t \ge 0$, then $u(t) \le d_2 E_q(-d_1 t^q)$, where $0 < q < 1$, $E_q(\cdot)$ is a Mittag-Leffler function with a parameter q.

Lemma 3. [18] If $r(t)$ is differentiable and $r'(t)$ is continuous, thus
$\frac{1}{2} D_t^q r^2(t) \le r(t) D_t^q r(t),\ 0 < q \le 1.$

Definition 5. [20] For $u(t) \in C(R)$, if $u(t+\omega) = -u(t)$ for $t \in R$, thus $u(t)$ is an anti-periodic function, where ω is a normal number.

Assumptions used in this article:

H_1 : $f_i(t)$ is bounded continuous excitation function and satisfies Lipschitz conditions, there exist $l_i > 0$, $\bar{f}_i > 0$ such $\left| f_i(\xi_1) - f_i(\xi_2) \right| \le l_i \left| \xi_1 - \xi_2 \right|$, $\left| f_i(t) \right| \le \bar{f}_i$, $\xi_1, \xi_2 \in R$, $i = 1, 2, \cdots n.$

H_2 : Excitation function $f_i(t)$ satisfies $f_i(u) = -f_i(-u)$, $u \in R$, $i = 1, 2, \cdots, n.$

H_3 : Input function $I_i(t)$ satisfies $I_i(t+\omega) = -I_i(t)$, $\left| I_i(t) \right| \le \bar{I}_i$, where $\omega > 0$, $\bar{I}_i \ge 0$, $i = 1, 2, \cdots, n.$

H_4 : Time-varying delays function $\tau_{ij}(t)$ is bounded, and differentiable, and satisfy $0 \le \dot{\tau}_{ij}(t) \le \tau^* < 1$, $t > 0$, $i = 1, 2, \cdots, n$.

3. Main results

Theorem 1. The solution of system (1) is bounded on $[0,T](0 \le T < +\infty)$ when H_1 and H_3 hold.
Proof. There is $D_t^{\alpha}|g(x)| \le \operatorname{sgn}(g(x))D_t^{\alpha}g(x)$ for a continuous function $g(x)$ and Definition 2. We get from (1):

$$D_t^{\alpha}|x_i(t)| \le -\beta_i|x_i(t)| + \sum_{j=1}^{n}|a_{ij}||f_j(x_j(t))| + \sum_{j=1}^{n}|b_{ij}||f_j(x_j(t-\tau_{ij}(t)))| + |I_i(t)|$$

$$\le -\beta_i|x_i(t)| + \sum_{j=1}^{n}(|a_{ij}| + |b_{ij}|)\overline{f}_j + \overline{I}_i. \tag{3}$$

Combined with Lemma 1, it can be deduced from (3):

$$|x_i(t)| \le -\beta_i D_t^{-\alpha}|x_i(t)| + D_t^{-\alpha}[\sum_{j=1}^{n}(|a_{ij}| + |b_{ij}|)\overline{f}_j + \overline{I}_i]$$

$$= -\beta_i D_t^{-\alpha}|x_i(t)| + [\sum_{j=1}^{n}(|a_{ij}| + |b_{ij}|)\overline{f}_j + \overline{I}_i]\frac{t^{\alpha}}{\Gamma(\alpha+1)}$$

$$\le -\beta_i D_t^{-\alpha}|x_i(t)| + [\sum_{j=1}^{n}(|a_{ij}| + |b_{ij}|)\overline{f}_j + \overline{I}_i]\frac{T^{\alpha}}{\Gamma(\alpha+1)}.$$

From Lemma 2:

$$|x_i(t)| \le \frac{[\sum_{j=1}^{n}(|a_{ij}| + |b_{ij}|)\overline{f}_j + \overline{I}_i]T^{\alpha}}{\Gamma(\alpha+1)}E_{\alpha}(-\beta_i t^{\alpha}),\ t \ge 0,\ i = 1,2,\cdots,n.$$

$$|x_i(t)| \le \frac{[\sum_{j=1}^{n}(|a_{ij}| + |b_{ij}|)\overline{f}_j + \overline{I}_i]T^{\alpha}}{\Gamma(\alpha+1)}E_{\alpha}(-\beta_i t^{\alpha}),\ t \ge 0,\ i = 1,2,\cdots,n.$$

That is the solution $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))^{\mathrm{T}}$ is bounded on $0 \le t \le T < +\infty$, where $E_{\alpha}(\cdot)$ is a Mittag-Leffler function with a parameter α.
Theorem 2. The solution of system (1) is Mittag-Leffler stabilization on $[0,T](T < +\infty)$, if

$$\eta = \min_{1\le i\le n}\{2\beta_i - \sum_{j=1}^{n}(|a_{ij}| + |b_{ij}|)\, l_j - \sum_{j=1}^{n}(|a_{ji}| + \frac{|b_{ji}|}{1-\tau^*})\, l_i\} > 0,$$

when H_1 and H_3 hold.
Proof. Suppose $x^*(t) = (x_1^*(t), x_2^*(t), \cdots, x_n^*(t))^{\mathrm{T}}$ and $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))^{\mathrm{T}}$ are the solutions of $x_i^*(s) = \varphi_i^*(s),\ D_t^{\alpha}x_i^*(s) = \psi_i^*(s)$ and $x_i(s) = \varphi_i(s),\ D_t^{\alpha}x_i(s) = \psi_i(s)$. Let $y_i(t) = x_i(t) - x_i^*(t)$, combined formula (1):

$$D_t^{\alpha}y_i(t) = -\beta_i y_i(t) + \sum_{j=1}^{n}a_{ij}[f_j(x_j(t)) - f_j(x_j^*(t))] + \sum_{j=1}^{n}b_{ij}[f_j(x_j(t-\tau_{ij}(t))) - f_j(x_j^*(t-\tau_{ij}(t)))]. \tag{4}$$

We get $D_t^{\alpha}y_i^2(t) \le 2y_i(t)D_t^{\alpha}y_i(t)$ from Lemma 3, and from (4):

$$D_t^{\alpha} y_i^2(t) \le 2y_i(t)\{-\beta_i y_i(t) + \sum_{j=1}^{n} a_{ij}[f_j(x_j(t)) - f_j(x_j^*(t))] + \sum_{j=1}^{n} b_{ij}[f_j(x_j(t-\tau_{ij}(t))) - f_j(x_j^*(t-\tau_{ij}(t)))]\}$$
$$\le -2\beta_i y_i^2(t) + 2|y_i(t)|[\sum_{j=1}^{n}|a_{ij}|l_j|y_j(t)| + \sum_{j=1}^{n}|b_{ij}|l_j|y_j(t-\tau_{ij}(t))|]$$
$$\le -2\beta_i y_i^2(t) + \sum_{j=1}^{n}|a_{ij}|l_j|y_i^2(t)+y_j^2(t)| + \sum_{j=1}^{n}|b_{ij}|l_j|y_i^2(t)+y_j^2(t-\tau_{ij}(t))|$$
$$=[-2\beta_i + \sum_{j=1}^{n}(|a_{ij}|+|b_{ij}|)l_j]y_i^2(t) + \sum_{j=1}^{n}|a_{ij}|l_j y_j^2(t) + \sum_{j=1}^{n}|b_{ij}|l_j y_j^2(t-\tau_{ij}(t)). \tag{5}$$

From (5):

$$\sum_{i=1}^{n} y_i^2(t) \le \sum_{i=1}^{n}[-2\beta_i + \sum_{j=1}^{n}(|a_{ij}|+|b_{ij}|)l_j + \sum_{j=1}^{n}|a_{ji}|l_i]\mathrm{D}_t^{-\alpha} y_i^2(t) + \sum_{i=1}^{n}\sum_{j=1}^{n}|b_{ji}|l_i\mathrm{D}_t^{-\alpha} y_i^2(t-\tau_{ij}(t)). \tag{6}$$

$t-\tau_{ij}(t)\in[-\tau_{ij}(t),0]$ when $t\in[0,\tau_{ij}(t)]$. Let $u=s-\tau_{ij}(s)$, then

$$D_t^{-\alpha} y_i^2(t-\tau_{ij}(t)) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} y_i^2(s-\tau_{ij}(s))\mathrm{d}s$$
$$= \frac{1}{\Gamma(\alpha)}\int_{-\tau_{ij}(0)}^{t-\tau_{ij}(t)} \frac{(t-u-\tau_{ij}(s))^{\alpha-1} y_i^2(u)}{1-\dot{\tau}_{ij}(s)}\mathrm{du}$$
$$\le \frac{\varphi_i^*}{(1-\tau^*)\Gamma(\alpha)}\int_{-\tau_{ij}(0)}^{t-\tau_{ij}(t)} (t-u-\tau_{ij}(t))^{\alpha-1}\mathrm{du}$$
$$= \frac{\varphi_i^*(t+\tau_{ij}(0)-\tau_{ij}(t))^{\alpha}}{(1-\tau^*)\Gamma(\alpha)\alpha}$$

$$\le \frac{\varphi_i^* T^{\alpha}}{(1-\tau^*)\Gamma(\alpha+1)}, \tag{7}$$

where $\varphi_i^* = \sup_{-\tau\le s\le 0}\{(\varphi_i^*(s)-\varphi_i(s))^2\},\ i=1,2,\cdots,n.$

$t-\tau_{ij}(t)\in[0,+\infty)$ when $t\in[\tau_{ij}(t),+\infty)$. Let $u=s-\tau_{ij}(s)$, then

$$D_t^{-\alpha} y_i^2(t-\tau_{ij}(t)) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} y_i^2(s-\tau_{ij}(s))\mathrm{d}s$$
$$= \frac{1}{\Gamma(\alpha)}\int_{-\tau_{ij}(0)}^{t-\tau_{ij}(t)} \frac{(t-u-\tau_{ij}(s))^{\alpha-1} y_i^2(u)}{1-\dot{\tau}_{ij}(s)}\mathrm{du}$$
$$= \frac{1}{\Gamma(\alpha)}[\int_{-\tau_{ij}(0)}^{0} \frac{(t-u-\tau_{ij}(s))^{\alpha-1} y_i^2(u)}{1-\dot{\tau}_{ij}(s)}\mathrm{du} + \int_0^{t-\tau_{ij}(t)} \frac{(t-u-\tau_{ij}(s))^{\alpha-1} y_i^2(u)}{1-\dot{\tau}_{ij}(s)}\mathrm{du}]$$
$$\le \frac{1}{(1-\tau^*)\Gamma(\alpha)}[\int_{-\tau_{ij}(0)}^{0} (-u)^{\alpha-1} y_i^2(u)\mathrm{du} + \int_0^{t-\tau_{ij}(t)} (t-u-\tau_{ij}(t))^{\alpha-1} y_i^2(u)\mathrm{du}]$$
$$\le \frac{1}{1-\tau^*}[\frac{\varphi_i^* T^{\alpha}}{\Gamma(\alpha+1)} + D_t^{-\alpha} y_i^2(t)], \tag{8}$$

where $\varphi_i^* = \sup_{-\tau \le s \le 0} \{(\varphi_i^*(s) - \varphi_i(s))^2\}$, $i = 1, 2, \cdots, n$.

We obtain from (7) and (8):

$$D_t^{-\alpha} y_i^2(t - \tau_{ij}(t)) \le \frac{1}{1-\tau^*}\left[\frac{\varphi_i^* T^\alpha}{\Gamma(\alpha+1)} + D_t^{-\alpha} y_i^2(t)\right],\ i = 1, 2, \cdots, n. \tag{9}$$

Substitute the result of (9) into (6):

$$\begin{aligned}\sum_{i=1}^n y_i^2(t) &\le \sum_{i=1}^n \Big[-2\beta_i + \sum_{j=1}^n (|a_{ij}| + |b_{ij}|) l_j + \sum_{j=1}^n (|a_{ji}| + \frac{|b_{ji}|}{1-\tau^*}) l_i\Big] \mathrm{D}_t^{-\alpha} y_i^2(t) + \sum_{i=1}^n \sum_{j=1}^n |b_{ji}| l_i \frac{\varphi_i^* T^\alpha}{(1-\tau^*)\Gamma(\alpha+1)} \\ &\le -\min_{1\le i\le n}\Big[2\beta_i - \sum_{j=1}^n (|a_{ij}| + |b_{ij}|) l_j - \sum_{j=1}^n (|a_{ji}| + \frac{|b_{ji}|}{1-\tau^*}) l_i\Big] \sum_{i=1}^n \mathrm{D}_t^{-\alpha} y_i^2(t) + \sum_{i=1}^n \sum_{j=1}^n |b_{ji}| l_i \frac{\varphi_i^* T^\alpha}{(1-\tau^*)\Gamma(\alpha+1)}. \end{aligned} \tag{10}$$

Combined with Lemma 2, it can be deduced from (10):

$$\|x - x^*\| = \sum_{i=1}^n (x - x^*)^2 \le M(\varphi - \varphi^*) E_\alpha(-\eta t^\alpha),\ t > 0, \tag{11}$$

where $M(\varphi - \varphi^*) = \sum_{i=1}^n \sum_{j=1}^n |b_{ji}| l_i \frac{\varphi_i^* T^\alpha}{(1-\tau^*)\Gamma(\alpha+1)}$,

$\eta = \min_{1\le i\le n}\{2\beta_i - \sum_{j=1}^n (|a_{ij}| + |b_{ij}|) l_j - \sum_{j=1}^n (|a_{ji}| + \frac{|b_{ji}|}{1-\tau^*}) l_i\} > 0$. Obviously $M(\varphi - \varphi^*) \ge 0$, and $M(0) = 0$, thus the solution of system (1) is Mittag-Leffler stabilization from Definition 4.

Theorem 3. System (1) has an anti-periodic solution when the Theorem 2 and H2hold, and the solution is Mittag-Leffler stabilization

Proof. For a positive integer *k* and a normal number *w* from H2 and H3, we obtain from (1):

$$\begin{aligned} D_t^\alpha[(-1)^{k+1} x_i(t + (k+1)\omega)] &= (-1)^{k+1}\Big[-\beta_i x_i(t + (k+1)\omega) + \sum_{j=1}^n a_{ij} f_j(x_j(t + (k+1)\omega)) \\ &\quad + \sum_{j=1}^n b_{ij} f_j(x_j(t + (k+1)\omega - \tau_{ij}(t))) + I_i(t + (k+1)\omega)\Big] \\ &= -\beta_i (-1)^{k+1} x_i(t + (k+1)\omega) + \sum_{j=1}^n a_{ij} f_j((-1)^{k+1} x_j(t + (k+1)\omega)) \\ &\quad + \sum_{j=1}^n b_{ij} f_j((-1)^{k+1} x_j(t + (k+1)\omega - \tau_{ij}(t))) + I_i(t),\ i = 1, 2, \cdots, n. \end{aligned} \tag{12}$$

So $(-1)^{k+1} x_i(t + (k+1)\omega)$ is the solution of system (1) for a positive integer k. $x(t)$ is bounded from Theorem 1, then there exists a positive constant N such that:

$$\left|(-1)^{k+1} x_i(t + (k+1)\omega)\right| \le N E_\alpha[-\eta(t + (k+1)\omega)^\alpha],\ i = 1, 2, \cdots, n.$$

Because $0 \le E_\alpha(-(\lambda t)^\alpha) \le 1, \lambda > 0$, so the sequence $\{(-1)^{k+1} x_i(t + k\omega)\}$ is equicontinuous and bounded uniformly. Reapplication Arzela-Ascoli theorem $\{(-1)^k x_i(t + k\omega)\}_{k\in N}$ converges to a continuous function $x_i^*(t)$ uniformly on any compact set in $[0, +\infty]$ by selecting a subsequence $\{k\omega\}_{k\in N}$, that is

$\lim_{k\to+\infty}(-1)^k x_i(t+k\omega)=x_i^*(t),\ i=1,2,\cdots,n.$

On the other hand, owing to

$x_i^*(t+\omega)=\lim_{k\to+\infty}(-1)^k x_i(t+\omega+k\omega)=-\lim_{k\to+\infty}(-1)^{k+1}x_i(t+(k+1)\omega)=-x_i^*(t),\ i=1,2,\cdots,n.$

So $x^*(t)$ is ω-anti-periodic function. Owing $(-1)^k x_i(t+k\omega)$ is the solution of system (1) for any $k\in N$, we obtain from (1):

$$D_t^{\alpha}[(-1)^k x_i(t+k\omega)]=-\beta_i(-1)^k x_i(t+k\omega)+\sum_{j=1}^{n}a_{ij}f_j((-1)^k x_j(t+k\omega))$$
$$+\sum_{j=1}^{n}b_{ij}f_j((-1)^k x_j(t+k\omega-\tau_{ij}(t)))+I_i(t),\quad i=1,2,\cdots,n.$$

We can continue to get when $f_i(\cdot)$ is continuous, then

$$\lim_{k\to+\infty}D_t^{\alpha}[(-1)^k x_i(t+k\omega)]=-\beta_i x_i^*(t)+\sum_{j=1}^{n}a_{ij}f_j(x_j^*(t))+\sum_{j=1}^{n}b_{ij}f_j(x_j^*(t+k\omega-\tau_{ij}(t)))+I_i(t),\quad i=1,2,\cdots,n.$$

So $x^*(t)$ is an anti-periodic solution of system (1). For any $x(t)$, the inequality holds from (11):

$$\left\|x(t)-x^*(t)\right\|=\sum_{i=1}^{n}\left|x_i(t)-x_i^*(t)\right|\le M(\varphi-\varphi^*)E_{\alpha}(-\eta t^{\alpha}),\ t>0,$$

so *x t()is an anti-periodic solution and Mittag-Leffler stabilization.

Remark: The stabilization and existence of anti-periodic solutions of nonlinear differential equations are of great significance in dynamic behavior, which plays a key role in physical phenomena [1–5]. The model of integer- order neural network system is a nonlinear differential equation, and fractional order neural network system is a generalization of integer-order neural network system, so fractional order neural network system is also a model of nonlinear differential equation generalization. From previous data, there are only discussions on the boundedness and asymptotic stabilization of almost periodic solution and w -periodic solution for fractional-order neural networks (e.g., [11–15]), we have not seen the results of authors exploring the dynamic behavior of the anti-periodic solution of a system. In the article, we mainly give the sufficient conditions for the existence of anti-periodic solutions and Mittag-Leffler stabilization of fractional order neural network systems. The results are new. This provides a new basis to further explore the dynamic properties of a system in theoretical research and practical application.

4. Numerical simulation

We consider fractional-order neural networks with time-varying delays:

$$D_t^{\alpha} x_i(t) = -\beta_i x_i(t) + \sum_{j=1}^{2} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{2} b_{ij} f_j(x_j(t-\tau_{ij}(t))) + I_i(t),\ i=1,2. \quad (13)$$

We get $\alpha = 0.85$, $\beta_1 = 1.85$, $\beta_2 = 1.9$, $a_{11} = \frac{1}{16}$, $a_{12} = \frac{1}{32}$, $a_{21} = -\frac{1}{16}$, $a_{22} = -\frac{1}{32}$, $b_{11} = -\frac{1}{16}$, $b_{12} = \frac{1}{32}$, $b_{21} = \frac{1}{16}$, $b_{22} = -\frac{1}{32}$, $f_i(x) = \frac{|x_i+1| - |x_i-1|}{50}$, $I_i(t) = \frac{\cos(8t)}{150}$, $\tau_{ij}(t) = \frac{2-e^{-t}}{3}$, therefore

$$I_i(t+\frac{\pi}{8}) = -I_i(t),\quad f_i(-x) = -\frac{|x_i+1| - |x_i-1|}{50} = -f_i(x),\ i=1,2.$$

Let $l_i = \frac{1}{25}$, $\bar{f}_i = \frac{1}{25}$, $\bar{I}_i = \frac{1}{150}$, $\tau^* = \frac{1}{3}$, $\omega = \frac{\pi}{8}$.

By calculating we have: $\eta = \min_{1\le i\le 2}\{2\beta_i - \sum_{j=1}^{2}(|a_{ij}| + |b_{ij}|)\, l_j - \sum_{j=1}^{2}(|a_{ji}| + \frac{|b_{ji}|}{1-\tau^*})\, l_i\} = 2.8884375 > 0$,

so Theorem 3 holds, the system (13) has a $\frac{\pi}{8}$-anti-periodic solution with Mittag-Leffler stabilization.

On the other hand, giving the transient change of $(x_1(t), y_1(t))$ and $(x_2(t), y_2(t))$ for system (13) by numerical simulation, as shown in the figures below (see Figures 1 and 2).

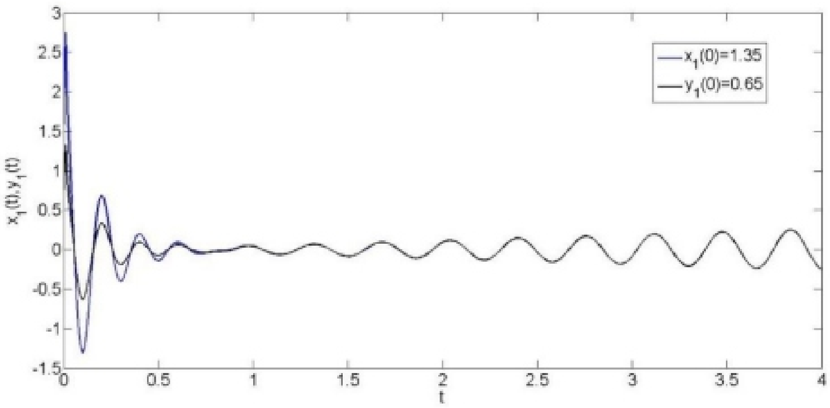

Figure 1. Transient change of $(x_1(t), y_1(t))$.

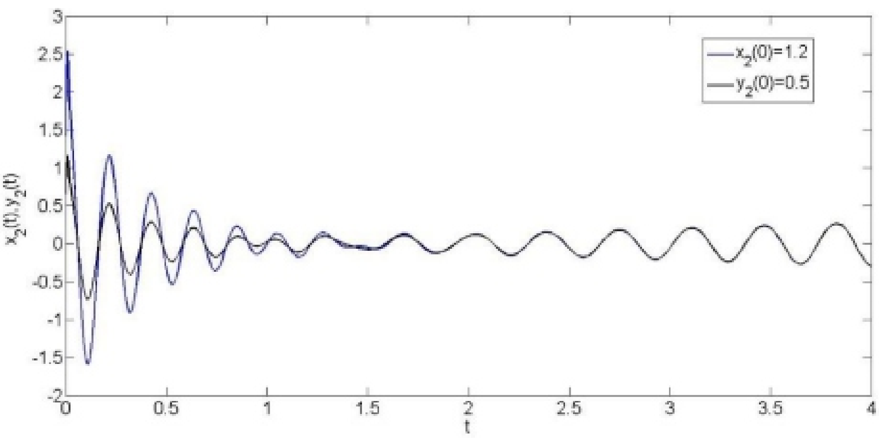

Figure 2. Transient change of $(x_2(t), y_2(t))$.

We gain a $\frac{\pi}{8}$-anti-periodic solution from the figures, it is consistent with the conclusion of theorems.

5. Conclusions

We study the dynamic behavior of fractional-order neural networks with time-varying delays in the article. Frist deriving the relationship between fractional-order integrals of state functions with and without time delay through the division of time interval and the properties of fractional-order calculus, the research method is innovative. Moreover, constructing the sequence solution of the system function which converges to a continuous function uniformly with the Arzela-Asoli theorem. In addition, giving the sufficient conditions the Mittag-Leffler stabilization, boundedness, and the existence of antiperiodic solutions for systems. Finally, the conclusion is feasible by a numerical simulation. Similarly, we can use the theoretical basis of this article to study the Mittag-Leffler stabilization of anti-periodic solutions of fractional-order Cohen-Grossberg neural networks and inertial Cohen-Grossberg neural networks, and so on.

Acknowledgments

Funding: Scientific Research Project of Shaoxing University Yuanpei College (No. KY2021C04).

Conflict of interest

The authors declare no conflict of interest.

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An alternating direction power-method for computing the largest singular value and singular vectors of a matrix

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A B S T R A C T

The singular value decomposition (SVD) is an important tool in matrix theory and numerical linear algebra. Research on the efficient numerical algorithms for computing the SVD ofa matrix is extensive in the past decades. In this paper, we propose an alternating direction power method for computing the largest singular value and singular vector of a matrix. The new method issimilar to the well-known power method but needs fewer operations in the iterations. Convergence of the new method is proved under suitable conditions. Theoretical analysis and numerical experiments show both that the new method is feasible and is effective than the power method in some cases.

Keywords: singular value decomposition (SVD); power method; alternating direction implicit (ADI); singular value; singular vector; convergence

1. Introduction

The singular value decomposition (SVD) is a key tool in matrix theory and numerical linear algebra, and plays an important role in many areas of scientific computing and engineering applications, such as least square problem [1], data mining [2], pattern recognition [3], image and signal processing [4,5], statistics, engineering, physics and so on (see [1–6]). Research on the efficient numerical methods for computing the singular values of a matrix has been a hot topic, many practical algorithms have been proposed for this problem. By using the symmetric QR method to A

TA, Golub and Kahan [7] presented an efficient algorithm named as GolubKahan SVD algorithm; Gu and Eisenstat [8] introduced a stable and efficient divide-and-conquer algorithm, called as Divide-and-Conquer algorithm as well as Bisection algorithm for computing the singular value decomposition (SVD) of a lower bidiagonal matrix, see also [1]; Drmac and Veselic [9] given the superior variant of the Jacobi algorithm and proposed a new one-sided Jacobi SVD algorithm for triangular matrices computed by revealing QR factorizations; many researchers such as Zha [10], Bojanczyk [11], Shirokov [12] and Novakovic [´ 13] came up with some methods for the problem of hyperbolic SVD; A Cross-Product Free (CPF) Jacobi-Davidson (JD) type method is proposed to compute a partial generalized singular value decomposition (GSVD) of a large matrix pair (A, B), which is referred to as the CPF-JDGSVD method [14]; many good references for these include [1, 2, 7–9, 15–26] and the references therein for details.

There are important relationships between the SVD of a matrix A and the Schur decompositions of the symmetric matrices *ATA, AAT* and $\begin{pmatrix} 0 & A^T \\ A & 0 \end{pmatrix}$. These connections to the symmetric eigenproblem allow us to adapt the mathematical and algorithmic developments of the eigenproblem to the singular value problem. So most of the algorithms mentioned above are analogs of algorithms for computing eigenvalues of symmetric matrices. All the algorithms mentioned above except for the Jacobi algorithm, had firstly to reduce A to bi-diagonal form. When the size of the matrix is large, this performance will become very costly. On the other hand, the Jacobi algorithm is rather slowly, though some modification has been added to it (see [9]). In some applications, such as the compressed sensing as well as the matrix completion problems [3, 27] or computing the 2-norm of a matrix, only a few singular values of a large matrix are required. In these cases, it is obvious that those methods, mentioned the above, for computing the SVD is not very suitable. If only the largest singular value and the singular vectors corresponding to the largest singular value of A is needed, the power method, which is used to approximate a largest eigenpair of an n × n symmetric matrix A, should be more suitable.Computing the largest singular value and corresponding singular vectors of a matrix is one of the most important algorithmic tasks underlying many applications including low-rank approximation, PCA, spectral clustering, dimensionality reduction, matrix completion and topic modeling. This paper consider the problem of computing the largest singular value and singular vectors corresponding to the largest singular value of a matrix. We propose an alternating direction method, a fast general purpose method for computing the largest singular vectors of a matrix when the target matrix can only be accessed through inaccurate matrix-vector products. In the other words, the proposed method is analogous to the well-known power method, but has much better numerical behaviour than the power method. Numerical experiments show that the new method is more effective than the power method in some cases.

The rest of the paper is organized as follows. Section 2 contains some notations and some general results that are used in subsequent sections. In Section 3 we propose the alternating direction powermethod in detail and give its convergence analysis. In Section 4, we use some experiments to show the effectiveness of the new method. Finally, we end the paper with a concluding remark in Section 5.

2. Preliminaries

The following are some notations and definitions we will use later.We use Rm×nto denote the set of all real m × n matrices, and Rnthe set of real n × 1 vectors. Thesymbol I denotes the n × n identity matrix. For a vector x ∈ Rn, kxk 2 denotes the 2-norm of x. For amatrix A ∈ Rn×n, AT is used to express the transpose of A, rank (A) is equal to the rank of a matrix A, kAk2 denotes the 2-norm of A and the Frobenius norm by kAkF is the maximum absolute value of the matrix entries of a matrix A. diag(a1, a2, . . . , an) represents the diagonal matrix with diagonal elementsa1, a2, . . . , an. If A ∈ R m×n , then there exist two orthogonal matrices

$$U = [u_1, u_2, \ldots, u_m] \in \mathbb{R}^{m\times m} \text{ and } V = [v_1, v_2, \ldots, v_n] \in \mathbb{R}^{n\times n}$$

such that

$$U^T AV = \begin{pmatrix} \Sigma_r & 0 \\ 0 & 0 \end{pmatrix}, \tag{2.1}$$

where $\Sigma_r = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_r)$, $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0, r = \text{rank}(A) \leqslant \min\{m, n\}$. The σ_i are the singular values of A and the vectors u_i and v_i are the ith left singular vector and the ith right singular vector respectively. And we have the SVD expansion

$$A = \sum_{i=1}^{r} \sigma_i u_i v_i^T.$$

This is the well-known singular value decomposition (SVD) theorem [2].

Lemma 2.1. (see Lemma 1.7 and Theorem 3.3 of [1]) Let $A \in \mathbb{R}^{m\times n}$. Then $\|A\|_2 = \|A^T\|_2 = \sqrt{\|AA^T\|_2} = \sigma_1$, where σ_1 is the largest singular value of A.

Lemma 2.2. (see Theorem 3.3 of [1]) Let $A \in \mathbb{R}^{m\times n}$ and $\sigma_i, u_i, v_i, i = 1, 2, \ldots, r$ be the singular values and the corresponding singular vectors of A respectively. Then

$$AA^T u_i = \sigma_i^2 u_i, \quad A^T A v_i = \sigma_i^2 v_i, \quad i = 1, 2, \ldots, r.$$

Lemma 2.3. (refer to Section 2.4 of [2] or Theorem 3.3 of [1])) Assume the matrix $A \in \mathbb{R}^{m\times n}$ has rank $r > k$ and the SVD of A be (2.1). The matrix approximation problem $\min_{\text{rank}(Z)=k} \|A - Z\|_F$ has the solution

$$Z = A_k = U_k \Sigma_k V_k^T,$$

where $U_k = (u_1, \ldots, u_k)$, $V_k = (v_1, \ldots, v_k)$ and $\Sigma_k = \text{diag}(\sigma_1, \ldots, \sigma_k)$.

Let $A \in \mathbb{R}^{n\times n}$. The power method for computing the module largest eigenvalue of A is as follows (see as Algorithm 4.1 of [1]).

Power method:
(1) Choose an initial vector $x_0 \in \mathbb{R}^n$. For $k = 0, 1, \ldots$ until convergence;
(2) Compute $y_{k+1} = Ax_k$;
(3) Compute $x_{k+1} = y_{k+1}/\|y_{k+1}\|_2$;
(4) Compute $\lambda_{k+1} = x_{k+1}^T A x_{k+1}$;
(5) Set $k = k + 1$ and go to (2).

The power method is very simple and easy to implement and is applied in many applications, for example, for the PCA problem (see [28]).

To compute the largest singular value and the corresponding singular vectors of A, we can apply the power method to AA^T or A^TA, without actually computing AA^T or A^TA.

Power method (for largest singular value):
(1) Choose an initial vector $u_0 \in \mathbb{R}^m$ and $v_0 \in \mathbb{R}^n$. For $k = 0, 1, \ldots$ until convergence;
(2) Compute $y_{k+1} = A(A^T u_k)$, $z_{k+1} = A^T(Av_k)$;
(3) Compute $u_{k+1} = y_{k+1}/\|y_{k+1}\|_2$, $v_{k+1} = z_{k+1}/\|z_{k+1}\|_2$;
(4) Compute $\lambda_{k+1} = u_{k+1}^T A u_{k+1}$ and $\sigma_{k+1} = \sqrt{\lambda_{k+1}}$;
(5) Set $k = k + 1$ and go to (2).

However, the power method will cost extra operations if the two singular vectors are needed and, in some cases, it converges very slowly. So, in the next section, we will propose a new iteration method for computing the largest singular value and the corresponding singular vectors of a matrix, which is similar to the power method but needs fewer operations in the iterations.

3. An alternating direction power-method

In this section, we will introduce an alternating direction power iteration method. The new methodis based on an important property of the SVD. From Lemma 2.3, it is known that the largest singular value σ1 and the corresponding singular vectors u1, v1 of A satisfy the following condition

$$\|A - \sigma_1 u_1 v_1^T\|_F = \min_{u,v} \|A - uv^T\|_F,$$

where $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$. Thus the problem of computing σ_1, u_1 and v_1 is equivalent to solve the optimization problem

$$\min_{u,v} \|A - uv^T\|_F.$$

Let $f(u, v) = \frac{1}{2}\|A - uv^T\|_F^2$. For the sake of simplicity, we will solve the equivalent optimization problem

$$\min_{u,v} f(u, v) = \min_{u,v} \frac{1}{2}\|A - uv^T\|_F^2, \tag{3.1}$$

where $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$. However, the problem (3.1) is difficult to solve since it is not convex for u and v. Fortunately, it is convex for u individually and so is v, so we can use the alternating direction method to solve it. Alternating minimization is widely used in optimization problems due to its simplicity, low memory and flexibility (see [20, 29]). In the following we apply an alternating method to solve the unconstrained optimization problem (3.1).

Suppose v_k was known, then (3.1) can be reduced to the unconstrained optimization problem

$$\min_{u} f(u, v_k) = \min_{u} \frac{1}{2}\|A - uv_k^T\|_F^2. \tag{3.2}$$

The (3.2) can be solved by many efficient methods, such as steepest decent method, Newton method, conjugate gradient (CG) method and so on (see [29]). Because Newton method is simple and converges fast, we will choose to use the Newton method. By direct calculation, we get

$$\nabla_u f = -(A - uv^T)v, \quad \Delta_u f = \|v\|_2^2 I.$$

Then applying Newton method, we get

$$\begin{aligned} u_{k+1} &= u_k - (\Delta_u f)^{-1} \nabla_u f \\ &= u_k + \frac{1}{\|v_k\|_2^2}(A - u_k v_k^T)v_k \\ &= \frac{1}{\|v_k\|_2^2} A v_k. \end{aligned}$$

Alternatively, when u_{k+1} is known, the problem (3.1) can be reduced to

$$\min_v f(u_{k+1}, v) = \min_v \frac{1}{2}\|A - u_{k+1}v^T\|_F^2. \tag{3.3}$$

Also,

$$\nabla_v f = -(A - uv^T)^T u, \quad \Delta_v f = \|u\|_2^2 I,$$

it is obtained that by applying Newton method

$$\begin{aligned} v_{k+1} &= v_k - (\Delta_v f)^{-1} \nabla_v f \\ &= v_k + \frac{1}{\|u_{k+1}\|_2^2}(A - u_{k+1} v_k^T)^T u_{k+1} \\ &= \frac{1}{\|u_{k+1}\|_2^2} A^T u_{k+1}. \end{aligned}$$

Solving (3.2) and (3.3) to high accuracy is both computationally expensive and of limited value if u_k and v_k are far from stationary points. So, in the following, we apply the two iterations alternately. Thus the alternating directional Newton method for solving (3.1) is

$$\begin{cases} u_{k+1} = & \frac{1}{\|v_k\|_2^2} A v_k \\ v_{k+1} = & \frac{1}{\|u_{k+1}\|_2^2} A^T u_{k+1} \end{cases}, \quad k = 0, 1, \ldots, \tag{3.4}$$

where $u_0 \in \mathbb{R}^m$ and $v_0 \in \mathbb{R}^n$ are both initial guesses. At each iteration, only two matrix-vector multiplications are required and the operation costs are about $4mn$, which is less than that of the power method.

Next, the convergence analysis of (3.4) would be provided.

Theorem 3.1 Let $A \in \mathbb{R}^{m \times n}$ and σ_1, u, v be the largest singular value and the corresponding singular vectors of A respectively. If $u_0 \in \mathbb{R}^m$ and $v_0 \in \mathbb{R}^n$ are both initial guesses such that the projections on u and v are not zero, then the iteration (3.4) is convergent with

$$\lim_{k \to \infty} \frac{u_k}{\|u_k\|_2} = u, \quad \lim_{k \to \infty} \frac{v_k}{\|v_k\|_2} = v,$$

and

$$\lim_{k \to \infty} \|u_k\|_2 \cdot \|v_k\|_2 = \sigma_1.$$

Proof. From (3.4) we can deduce that

$$u_{k+1} = \frac{1}{||v_k||_2^2} A v_k = \frac{||u_k||_2^2}{||A^T u_k||_2^2} A A^T u_k,$$

$$v_{k+1} = \frac{1}{||u_{k+1}||_2^2} A^T u_{k+1} = \frac{||v_k||_2^2}{||A v_k||_2^2} A^T A v_k.$$

As in the proof of the power method (see as [2]), if the projections of u_0 on u, and v_0 on v are not zero, then we have

$$\lim_{k\to\infty} \frac{u_k}{||u_k||_2} = u, \quad \lim_{k\to\infty} \frac{v_k}{||v_k||_2} = v.$$

On the other hand, we have

$$\begin{aligned} ||u_{k+1}||_2 \cdot ||v_{k+1}||_2 =& \frac{1}{||v_k||_2^2} ||A v_k||_2 \frac{||v_k||_2^2}{||A v_k||_2^2} ||A A^T u_k||_2 \\ =& \frac{1}{||A v_k||_2} ||A A^T u_k||_2 \\ \le& ||A^T||_2 \\ =& \sigma_1. \end{aligned}$$

Thus the sequence $\{||u_k||_2 \cdot ||v_k||_2\}$ is bounded from the above. By

$$||u_{k+1}||_2 \cdot ||v_{k+1}||_2 = \frac{1}{||v_k||_2^2} ||A v_k||_2 \frac{1}{||u_{k+1}||_2^2} ||A^T u_{k+1}||_2,$$

we can conclude that when $k \to \infty$

$$\begin{aligned} ||u_{k+1}||_2^2 \cdot ||v_{k+1}||_2 \cdot ||v_k||_2 =& ||A \frac{1}{||v_k||_2} v_k||_2 \cdot ||A^T \frac{1}{||u_{k+1}||_2} u_{k+1}||_2 \\ \to& ||A v||_2 \cdot ||A^T u||_2 \\ =& \sigma_1^2. \end{aligned}$$

Therefore,

$$\lim_{k\to\infty} ||u_k||_2 \cdot ||v_k||_2 = \sigma_1.$$

□

We now propose the alternating direction power-method with the discussions above for computing the largest singular value and corresponding singular vectors of a matrix as follows.

Alternating Direction Power-Method (ADPM):
(1) Choose initial vectors $v_0 \in \mathbb{R}^n$. For $k = 0, 1, \ldots$ until convergence;
(2) Compute $u_{k+1} = A v_k / ||v_k||_2^2$;
(3) Compute $v_{k+1} = A^T u_{k+1} / ||u_{k+1}||_2^2$;
(4) Compute $\mu_{k+1} = ||u_{k+1}||_2 \cdot ||v_{k+1}||_2$;
(5) Set $k = k + 1$ and go to (2).

4. Numerical experiments

Here, we use several examples to show the effectiveness of the alternating direction power-method (ADPM). We compare ADPM with the power method (PM) and present numerical results in terms of the numbers of iterations (IT), CPU time (CPU) in seconds and the residue (RES), where the measurement method of CPU time in seconds is uniformly averages over multiple runs by embed matlab functions tic/toc at each iteration step and

$$\text{RES} = \text{abs}\left(\|u_{k+1}\|_2 \cdot \|v_{k+1}\|_2 - \|u_k\|_2 \cdot \|v_k\|_2\right).$$

The initial vectors u0 and v0 are chosen randomly by the matlab statements u0 = rand(m, 1) and v0 = rand(n, 1). In our implementations all iterations are performed in matlab (R2016a) on the same workstation with an Intel® Core(TM) i7-6700 CPU @ 3.40GHz that has 16GB memory and 32-bit operating system, and are terminated when the current iterate satisfies RES< e −12 or the number of iterations is more than 9000, which is denoted by '-'.

Experiment 4.1. In the first experiment, we generate random matrices with uniformly distributed elements by the matlab statement

A = rand(m, n).

For different sizes of m and n, we apply the power method and the alternating direction power-method with numerical results reported in Table 1.

Table 1. Numerical results of Experiment 4.1.

m	n	Method	IT	CPU	RES	RATIO(%)
500	100	PM	7	0.001340	1.4211e-13	
		ADPM	4	0.000471	1.4172e-15	35.15
500	200	PM	7	0.002753	5.6843e-14	
		ADPM	4	0.000641	2.8422e-14	23.28
1000	200	PM	7	0.015660	2.8422e-14	
		ADPM	4	0.003914	1.7053e-13	24.99
1000	500	PM	6	0.017024	2.8422e-13	
		ADPM	3	0.004934	2.8422e-13	28.98
2000	500	PM	6	0.030742	5.6843e-14	
		ADPM	3	0.016660	1.1369e-13	54.19
2000	1000	PM	6	0.060125	4.5475e-13	
		ADPM	3	0.014466	1.1369e-13	24.05

Experiment 4.2. In this experiment, we generate random matrices with normally distributed elements by A = randn(m, n).

For different sizes of m and n, we apply the power method and the alternating direction power-methodto A. Numerical results are reported in Table 2.

Table 2. Numerical results of Experiment 4.2.

m	n	Method	IT	CPU	RES	RATIO(%)
500	100	PM	711	0.084958	9.2371e-13	
		ADPM	372	0.046376	9.6634e-13	54.59
500	200	PM	652	0.084715	8.8818e-13	
		ADPM	449	0.058401	9.3081e-13	68.94
1000	200	PM	1124	0.159432	9.9476e-13	
		ADPM	700	0.121118	9.7344e-13	75.96
1000	500	PM	1288	1.122289	9.8055e-13	
		ADPM	782	0.378765	9.8765e-13	33.75
2000	500	PM	1744	3.627609	9.6634e-13	
		ADPM	961	1.221796	9.0949e-13	33.68
2000	1000	PM	4364	20.032671	9.9476e-13	
		ADPM	2377	5.696365	9.0949e-13	28.42

Experiment 4.3. In this experiment, we use some test matrices with size n × n from the university of Florida sparse matrix collection [30]. Numerical results are reported in Table 3.

Table 3. Numerical results of Experiment 4.3.

Matrix	size	Method	IT	CPU	RES	RATIO(%)
lshp1009	1009	PM	1488	0.064107	9.9298e-13	
		ADPM	745	0.019718	9.9654e-13	30.76
dwt_1005	1035	PM	40	0.005109	7.2120e-13	
		ADPM	35	0.003844	6.2172e-13	75.24
bcsstk13	2003	PM	1519	0.511403	9.7877e-13	
		ADPM	842	0.174025	9.9920e-13	34.03
dwt_2680	2680	PM	514	0.070507	1.4172e-09	
		ADPM	239	0.046361	9.7167e-13	65.75
rw5151	5151	PM	1006	0.128023	9.8632e-13	
		ADPM	590	0.057257	9.7056e-13	44.72
g7jac040	11790	PM	26	0.038169	0	
		ADPM	17	0.012170	0	31.88
epb1	14734	PM	-	-	-	-
		ADPM	6132	1.541585	9.9926e-13	-

In particular, compared with the cost of the power method, we can find that the cost of the alternating direction power-method is discounted, up to 23.28%. The "ratio", defined in the following, in the Tables 1–3 can show this effectiveness.

$$\text{RATIO} = \frac{\text{the CPU of the ADPM}}{\text{the CPU of the PM}} \times 100\%.$$

In order to show numerical behave of two methods, the cost curves of the methods are clearly given, which are shown in Figures 1–3.

From Tables 1–3, we can conclude that ADPM needs fewer iterations and less CPU time than the power method. So it is feasible and is effective in some cases.

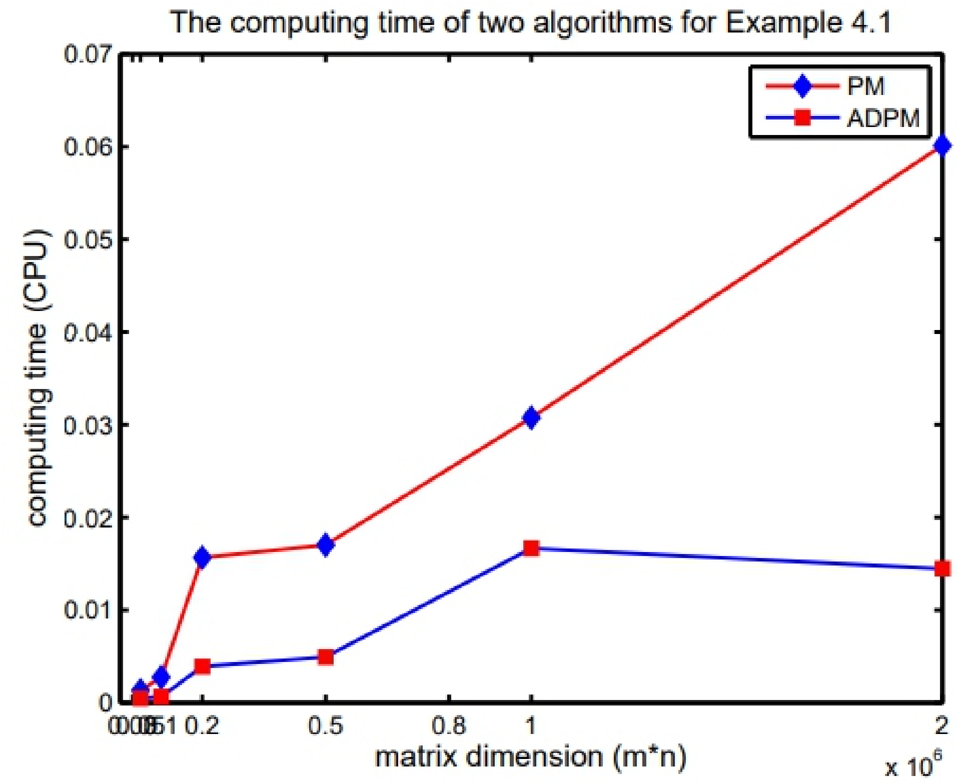

Figure 1. Comparison curve of the PM and ADPM methods for Example 4.1.

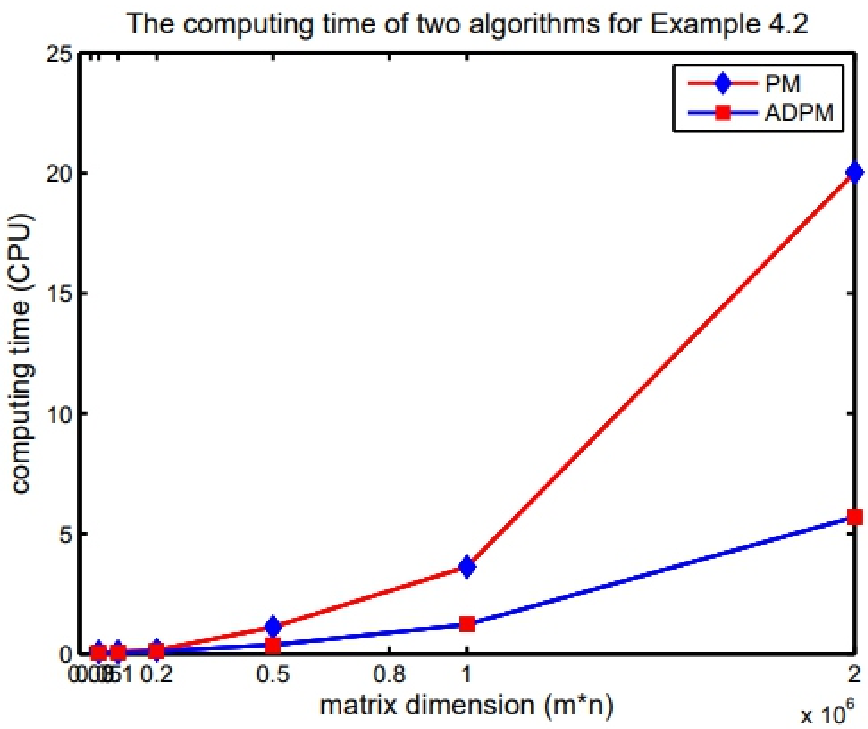

Figure 2. Comparison curve of the PM and ADPM methods for Example 4.2.

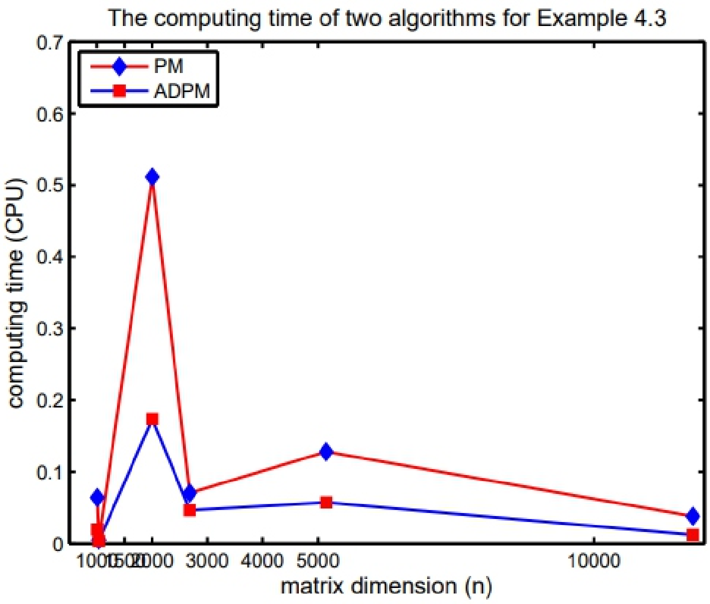

Figure 3. Comparison curve of the PM and ADPM methods for Example 4.3.

5. Conclusions

In this study, we have proposed an alternating direction power-method for computing the largest singular value and singular vector of a matrix, which is analogs to the power method but needs fewer operations in the iterations since using the technique of alternating. Convergence of the alternating direction power-method is proved under suitable conditions. Numerical experiments have shown that the alternating direction power-method is feasible and more effective than the power method in some cases.

Acknowledgments

The authors are very much indebted to the anonymous referees for their helpful comments and suggestions which greatly improved the original manuscript of this paper. The authors are so thankful for the support from the NSF of Shanxi Province (201901D211423) and the scientific research project of Taiyuan University, China (21TYKY02).

Conflict of interest

The authors declare that they have no conflict of interests.

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Positive solution for a class of nonlinear fourth-order boundary value problem

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ABSTRACT

In this paper, we are concerned with the existence of positive solutions for boundary value problems of nonlinear fourth-order differential equations

$$u^{(4)} + c(x)u = \lambda a(x)f(u), \quad x \in (0,1),$$
$$u(0) = u(1) = u''(0) = u''(1) = 0,$$

where a(x) may change signs. The proof of main results is based on Leray-Schauder's fixed point and the properties of Green's function of the fourth-order differential operator Lcu = u(4) +c(x)u.

Keywords: fourth-order differential operator; positive solution; boundary value problem;
Leray-Schauder's fixed point theorem
Mathematics Subject Classification: *34A08, 34B15, 35J05*

Introduction

Nonlinear mathematical models [1,2] were widely used in many fields. In particular, boundary value problems of nonlinear differential equations have received extensive attention and have been intensivelystudied in the past thirty years, see [3, 4]. We point out that boundary value problems for second orderdifferential equations, see, for example [5–9] and the references therein. While studies about boundaryvalue problems of nonlinear fourth-order differential equations are much more less. One of the earliestpapers about boundary value problems of nonlinear fourth-order differential operator is [10] from R.Ma and H. Wang, there they concerned the following problem

$$y^{(4)} - h(x)f(y(x)) = 0, \quad x \in (0,1)$$

with boundary condition

$$y(0) = y(1) = y''(0) = y''(1) = 0$$

$$y(0) = y'(1) = y''(0) = y'''(1) = 0.$$

By the fixed point theorem in cone, they proved the existence of positive solutions under the conditions that f is either superlinear or sublinear. In another paper [11], the author obtained the positive solution of the following problem

$$u^{(4)} + \beta u'' - \alpha u = f(t,u), \quad x \in (0,1)$$
$$u(0) = u(1) = u''(0) = u''(1) = 0$$

by the fixed point theorem in cone. R.Vrabel [12] studied the upper solution and lower solution of the problem

$$y^{(4)}(x) + \lambda y''(x) = h(x, y(x)), \quad x \in (0,1)$$
$$y(0) = y(1) = y''(0) = y''(1) = 0.$$

There are many other papers we will not list but we find that they have a common point, that is, the fourth-order differential operators they dealt with can be resolved into composition of two secondorder positive linear operators. And therefore, the corresponding Green's function for fourth-order.

linear operator is the form of the product of two Green's functions for second-order linear operators.
In a recently paper [13], Drabet discussed the existence of positive solutions for the following ´ fourth-order linear problem

$$u^{(4)} + c(x)u = h(x), \quad x \in (0,1),$$
$$u(0) = u(1) = u''(0) = u''(1) = 0.$$

Obviously, the fourth-order differential operator can not be resolution into composition of two second-order positive linear operators. For more results on nonlinear fourth-order differential operator problems we can refer to [14, 15].
Based on the above literature inspiration. We now consider the fourth-order nonlinear equation with Dirichlet boundary conditions

$$u^{(4)} + c(x)u = \lambda a(x) f(u), \tag{1.1}$$

$$u(0) = u(1) = u''(0) = u''(1) = 0, \tag{1.2}$$

where c(x), a(x) satisfy some conditions that we will give bellow, especially, a(x) may change signs.
We make the following assumptions throughout the paper:

(A1) $-\pi^4 < c(x) < c_0$,
(A2) $f : \mathbb{R}^+ \to \mathbb{R}$ is continuous, and $f(0) > 0$,
(A3) $a : [0, 1] \to \mathbb{R}$ is continuous with $a(x) \not\equiv 0$, and there exists a constant $K > 0$ such that

$$\int_0^1 G(x, y)a^+(y)dy \geq K \int_0^1 G(x, y)a^-(y)dy$$

for every $x \in (0, 1)$, where a^+(resp. a^-) is the positive (resp. negative) part of a, c_0 is the constant given in [13], and $G(x, y)$ is the Green's function of L_c with boundary conditions (1.2).

Our main result is as follows:

Theorem 1.1. *Let (A1)–(A3) hold. Then there exists a positive number λ^* such that* (1.1) *and* (1.2) *have a positive solution for* $0 < \lambda < \lambda^*$.

Remark 1.1. *Since the fourth-order differential operator can not be resolution into composition of two second-order positive linear operators, as a result, the Green's function have no explicit expression. So the method or technic used in [10–12] does not work. To deal with the new case and the difficult it brings, we are inspired by the method to second-order elliptic boundary value problems in [8], and the result that the fourth-order operator $u^{(4)} + c(x)u$ is strictly inverse positive in [13, 16]. Thanks to the existence and its properties of the Green's function given in [17–19], we obtain the existence of a positive solution to the problems* (1.1) *and* (1.2).

2. Preliminaries

In this section we present two important lemmas. The main method we use is the fixed point theorem of Leray-Schauder type. We refer interested readers to the literature [20, 21].

Set

$$W = \{u \in C^4([0, 1]) : u(0) = u(1) = u''(0) = u''(1) = 0\},$$

and let the linear operator $L_c : W \to C([0, 1])$ defined by

$$L_c u = u^{(4)} + c(x)u.$$

Then the boundary value problems (1.1) and (1.2) are equivalent to the operator equation

$$L_c u = \lambda a(x) f(u).$$

Lemma 2.1. *Let (A1) hold. Then L_c is strictly inverse positive, and therefore it has a positive Green's function.*

Proof. L_c is strictly inverse positive, we can refer to [13, 16] and the reference therein. From the definition of L_c is strictly inverse positive there and the well-known truth that

$$L_c u = h(x)$$

is equivalent to

$$u(x) = \int_0^1 G(x, y)h(y))dy,$$

we can get the positiveness of the Green's function $G(x, y)$ immediately.

Lemma 2.2. *Let (A1)–(A3) hold, and let $0 < \delta < 1$. Then there exists a positive number $\bar{\lambda}$ such that, for $0 < \lambda < \bar{\lambda}$, the problem*

$$u^{(4)} + c(x)u = \lambda a(x)^{+} f(u) \tag{2.1}$$

$$u(0) = u(1) = u''(0) = u''(1) = 0 \tag{2.2}$$

has a positive solution $\tilde{u}_\lambda$ with $|\tilde{u}_\lambda| \to 0$ as $\lambda \to 0$, and

$$\tilde{u}_\lambda(x) \geq \lambda\delta f(0)p(x), \quad x \in (0, 1),$$

where $p(x) = \int_0^1 G(x, y)a^{+}(y)dy$.

Proof. It follows from Lemma 2.1 that L_c is strictly inverse positive, and therefore it has a positive Green's function $G(x, y)$. For each $u \in C([0, 1])$, let

$$Au(x) = \lambda \int_0^1 G(x, y)a^{+}(y)f(u(y))dy, \quad x \in [0, 1].$$

Then the fixed points of A are solutions of problems (2.1) and (2.2). We now verify the condition of Leray-Schauder fixed point theorem to show that A has a fixed point for λ small.

Firstly, $A : C([0, 1]) \to C([0, 1])$ is completely continuous by the assumptions and Arzela-Ascoli theorem.

Secondly, we find a bounded open set Ω with $0 \in \Omega$ in $C([0, 1])$, such that for $u \in C(\bar{\Omega})$ and $\theta \in (0, 1)$, if $u = \theta Au$, then $u\bar{\in}\partial\Omega$.

By $(A2)$, the function $g(s) = \frac{f(s)}{f(0)}$ is continuous and $g(0) = 1$, since $0 < \delta < 1$, we can choose $\varepsilon > 0$ such that

$$f(s) > \delta f(0) \quad s \in [0, \varepsilon].$$

Also we have

$$|Au|_0 \leq \lambda|p|_0\bar{f}(|u|_0) \leq \lambda|p|_0\bar{f}(\varepsilon), \quad u \in [0, \varepsilon],$$

$$|Au|_0 \leq \lambda|p|_0\bar{f}(|u|_0) \leq \lambda|p|_0\bar{f}(\varepsilon), \quad u \in [0, \varepsilon],$$

where $\bar{f}(t) = \max_{0\leq s\leq t} f(s)$, and $|\cdot|_0$ is the usual norm in $C([0, 1])$.

Suppose $\lambda < \frac{1}{2|p|_0\bar{f}(\varepsilon)} =: \bar{\lambda}$, then there exists a $A_\lambda \in (0, \varepsilon)$ such that

$$\frac{\bar{f}(A_\lambda)}{A_\lambda} = \frac{1}{2\lambda|p|_0}.$$

Let $\Omega = \{u \in C([0, 1]) : |u|_0 < A_\lambda\}$ and $\theta \in (0, 1)$ such that $u = \theta Au$. Then we have

$$|u|_0 \leq |Au|_0 \leq \lambda|p|_0\bar{f}(|u|_0),$$

or

$$\frac{\bar{f}(|u|_0)}{|u|_0} \geq \frac{1}{\lambda|p|_0}.$$

So $u \neq A_\lambda$, which means $u\bar{\in}\partial\Omega$.

By the Leray-Schauder fixed point theorem, A has a fixed point $\tilde{u}_\lambda$ in Ω for $0 < \lambda < \bar{\lambda}$, that is, problems (2.1) and (2.2) have a positive solution $\tilde{u}_\lambda$ with $\tilde{u}_\lambda \leq A_\lambda < \varepsilon$. Notice that $A_\lambda \to 0$ as $\lambda \to 0$, $|\tilde{u}_\lambda| \to 0$ as $\lambda \to 0$ and

$$\tilde{u}_\lambda(x) = A\tilde{u}_\lambda(x) \geq \lambda\delta f(0)p(x), \quad x \in (0, 1).$$

The proof is completed.

3. Proof of Theorem 1.1

Proof of Theorem 1.1. Let $q(x) = \int_0^1 G(x, y)a^-(y)dy$, recall that $p(x) = \int_0^1 G(x, y)a^+(y)dy$. By $(A2)$,

$$q(x) \leq \frac{1}{K}p(x).$$

From the proof of Lemma 2.2 that $g(0) = 1$, there is a $\alpha \in (0, 1)$ and we can choose $1 < \sigma < K$, such that $f(s) < \sigma f(0)$, and $\gamma = \frac{\sigma}{K} \in (0, 1)$, then we have

$$q(x)f(s) \leq \gamma f(0)p(x) \tag{3.1}$$

for $s \in [0, \alpha]$, $x \in (0, 1)$. Fix $\delta \in (0, 1)$ and let $\lambda^* > 0$ be such that

$$|\tilde{u}_\lambda|_0 + \lambda\delta f(0)|p|_0 \leq \alpha \tag{3.2}$$

for $0 < \lambda < \lambda^*$, where $\tilde{u}_\lambda$ is the solution of (2.1) and (2.2) given by Lemma 2.2, and

$$|f(s) - f(t)| \leq \frac{\delta - \gamma}{2} \cdot f(0) \tag{3.3}$$

for $s,\ t \in [-\alpha, \alpha]$ with $|s - t| \leq \lambda^*\delta f(0)|p|_0$.

Let $0 < \lambda < \lambda^*$, we look for a solution $u_\lambda = \tilde{u}_\lambda + v_\lambda$. Since $\tilde{u}_\lambda$ is the solution of (2.1) and (2.2), then v_λ solves

$$L_c v_\lambda = \lambda a^+[f(\tilde{u}_\lambda + v_\lambda) - f(\tilde{u}_\lambda)] - \lambda a^- f(\tilde{u}_\lambda + v_\lambda), \quad x \in (0, 1),$$
$$v_\lambda(0) = v_\lambda(1) = v_\lambda''(0) = v_\lambda''(1) = 0.$$

For each $w \in C([0, 1])$, let $v = Aw$ be the solution of

$$L_c v = \lambda a^+[f(\tilde{u}_\lambda + w) - f(\tilde{u}_\lambda)] - \lambda a^- f(\tilde{u}_\lambda + w), \quad x \in (0, 1),$$

$$v(0) = v(1) = v''(0) = v''(1) = 0,$$

where the operator A is as in Lemma 2.2, we have

$$Aw(x) \ = \ \lambda \int_0^1 G(x, y)a^+(y)[f(\tilde{u}_\lambda(y) + w(y)) - f(\tilde{u}_\lambda(y))]dy$$

$$-\lambda \int_0^1 G(x,y)a^-(y)f(\tilde{u}_\lambda(y)+w(y))dy, \quad x \in [0,1],$$

and A is completely continuous.

Let

$$\bar{\Omega}' = \{v \in C([0,1]); |v|_0 \le \lambda\delta f(0)|p|_0\},$$

if $v \in C(\bar{\Omega}')$ and $\theta \in (0,1)$, such that $v = \theta A v$, that is

$$\begin{aligned} v(x) &= \lambda\theta \int_0^1 G(x,y)a^+(y)[f(\tilde{u}_\lambda(y)+v(y)) - f(\tilde{u}_\lambda(y))]dy \\ &\quad -\lambda\theta \int_0^1 G(x,y)a^-(y)f(\tilde{u}_\lambda(y)+v(y))dy, \quad x \in [0,1], \end{aligned}$$

we are going to show that

$$|v|_0 \neq \lambda\delta f(0)|p|_0.$$

Suppose the contrary that $|v|_0 = \lambda\delta f(0)|p|_0$. Then by (3.2) and (3.3), we get

$$|\tilde{u}_\lambda + v|_0 \le |\tilde{u}_\lambda|_0 + |v|_0 \le \alpha,$$

and

$$|f(\tilde{u}_\lambda + v) - f(\tilde{u}_\lambda)|_0 \le \frac{\delta-\gamma}{2} \cdot f(0),$$

together with (3.1) implies that

$$\begin{aligned} |v(x)| &\le \lambda \cdot \frac{\delta-\gamma}{2} \cdot f(0)p(x) + \lambda\gamma f(0)p(x) \\ &= \lambda \cdot \frac{\delta+\gamma}{2} \cdot f(0)p(x), \qquad x \in [0,1], \end{aligned} \tag{3.4}$$

and

$$|v|_0 \le \lambda \cdot \frac{\delta+\gamma}{2} \cdot f(0)|p|_0 < \lambda\delta f(0)|p|_0,$$

a contradiction.

By the Leray-Schauder fixed point theorem, A has a fixed point v_λ in $\bar{\Omega}'$ with $|v_\lambda|_0 \le \lambda\delta f(0)|p|_0$. Hence v_λ satisfies (3.4), and using Lemma 2.2, we obtain

$$\begin{aligned} u_\lambda(x) &= \tilde{u}_\lambda(x) + v_\lambda(x) \ge \tilde{u}_\lambda(x) - |v_\lambda(x)| \\ &\ge \lambda\delta f(0)p(x) - \lambda \cdot \frac{\delta+\gamma}{2} \cdot f(0)p(x) = \lambda \cdot \frac{\delta-\gamma}{2} \cdot f(0)p(x) > 0. \end{aligned}$$

We have proved that uλ is a positive solution of (1.1) and (1.2).

4. Conclusions

In this paper, we mainly study the existence of solutions to a class of nonlinear fourth-order Dilrichlet boundary value problems through Leray-Schauder's fixed point theorem, and show theasymptotic behavior of the solution as λ changes. In the future, we can try to construct such solutions,give the properties of the solutions, or study numerical solutions for such problems.

Acknowledgments

The authors would like to thank the referees and editors for providing very helpful comments and suggestions. This work is supported by Natural Science Foundation of Gansu Province (No. 21JR1RA317) and Young Doctor Fund project of Gansu Province (No. 2022QB-173).

Conflict of interest

The authors declare no conflicts of interest in this paper.

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Destruction of solutions for class of wave p(x)−bi-Laplace equation with nonlinear dissipation

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A B S T R A C T

An initial value problem is considered for the nonlinear dissipative wave equation containing the p(x)-bi-Laplacian operator. For this problem, sufficient conditions for the blow-up withnonpositive initial energy of a generalized solution are obtained in finite time where a wide variety of techniques are used.

Keywords: Sobolev spaces with variable exponents; p(x)-bi-Laplace; blow-up; initial energy

1. Introduction and formulation of the problem

Let Ω is a bounded open set in $\mathbb{R}^n$ with a sufficiently smooth boundary $\partial\Omega$ and outward facing unit normal n, let $u(x,t) = u$. The purpose of this study is to obtain sufficient conditions to prove the global nonexistence result for initial boundary value problem of wave equation containing the $p(x)$-bi-Laplacian operator

$$\begin{cases} \partial_{tt}u + \Delta_x(div(|\Delta_x u|^{p(x)-2}\nabla_x u)) + \mu|\partial_t u|^{m-2}\partial_t u = b|u|^{r-2}u, & x \in \Omega, t > 0 \\ u = \Delta_x u = 0, & x \in \partial\Omega, t > 0 \\ u = u_0(x) \in \mathcal{V}(\Omega), \quad \partial_t u = u_1(x) \in L^{p(x)}(\Omega), & x \in \Omega, t = 0, \end{cases} \tag{1.1}$$

where μ, b are positive constants, the spaces $\mathcal{V}(\Omega)$ and $L^{p(x)}(\Omega)$ are defined in Definition 1 and (2.1).

This problem is a mathematical model of wave processes in mathematical physics, taking into account dissipation and the relationship between the different parameters. Recently, new strongly nonlinear dissipative wave equations of the hyperbolic type have been intensively considered in mathematical physics. It should be mentioned that many authors have studied the question of existence, uniqueness, regularity and blow-up of weak solutions for parabolic and elliptic equations involving the p(x)-Laplacian view of its applications in the fields of nonlinear elasticity, fluid dynamics, elastic mechanics etc, see [4, 6, 8, 12, 13, 15–17, 20, 21] and the references therein.

In [2], the author established the existence of weak solutions for p(x, t)-Laplacian equation with damping term

$$\partial_{tt}u = div(a(x,t)|\nabla_x u|^{p(x,t)-2}\nabla_x u) + \alpha\Delta_x u + b(x,t)u|u|^{\sigma(x,t)-2} + f(x,t),$$

and proved the blow-up of weak solutions with negative initial energy, where α is a nonnegative constant and a, b, p, σ are given functions. Such equations are usually referred as equations with nonstandard growth conditions. It is proved the blow-up result of weak solutions with negative initial energy as well as for certain solutions with positive initial energy to the following equation

$$\partial_{tt}u - div(|\nabla_x u|^{r(.)-2}\nabla_x u) + a\partial_t u|\partial_t u|^{m(.)-2} = bu|u|^{p(.)-2},$$

In particular case $p(x) = 2$, the problem (1.1) is reduced to the Petrovsky type equation

$$\begin{cases} \partial_{tt}u + \Delta_x^2 u + \mu|\partial_t u|^{m-2}\partial_t u = b|u|^{r-2}u \\ u = \frac{\partial u}{\partial n} = 0 \\ u(x,0) = u_0(x), \quad \partial_t u(x,0) = u_1(x). \end{cases}$$

It is studied where, the author established an existence result and proved that the solution continues to exist globally if $m \geq r$ and blows up in finite time if $m < r$ and the initial energy is negative. Motivated by the above work, we obtain the blow-up results of solution to problem (1.1) for nonpositive initial energy. In order to state our result, we use some ideas introduced in the work of [7, 11, 14].

2. Main results

In this section, we recall some definitions and basic properties about the generalized Sobolev and Lebesgue spaces with variable exponents. The reader is referred to [3, 5, 9, 10] for more detailes.
Denote

$$C_+(\bar{\Omega}) = \{p(x) : p(x) \in C(\bar{\Omega}), p(x) > 2, for\ all\ x \in \bar{\Omega}\},$$

and

$$p^- = \operatorname{ess}\inf_{x\in\bar{\Omega}} p(x),\ \ p^+ = \operatorname{ess}\sup_{x\in\bar{\Omega}} p(x).$$

Then, the mesurable function

$$p : \bar{\Omega} \to [p^-, p^+] \subset (2, \infty),$$

satisfies the log-Hölder continuity condition

$$|p(x) - p(y)| \leq \frac{C}{\ln(e + |x-y|^{-1})} \quad , for\ all\ x, y \in \Omega.$$

For some $\lambda > 0$ the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is defined as the set of mesurable functions $u : \Omega \to \mathbb{R}$ such that $\mathcal{P}_{p(.)}(\lambda u) < \infty$ with respect to the Luxemburg norm

$$\|u\|_{p(x)} = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx < \infty\right\}, \tag{2.1}$$

where

$$\mathcal{P}_{p(.)}(u) = \int_\Omega |u(x)|^{p(x)}dx, \quad \|u\|_{p(x)} := \|u\|_{L^{p(x)}(\Omega)}.$$

The space $(L^{p(x)}(\Omega), \|.\|_{p(x)})$ is separable, uniformly convex, reflexive and its dual space is $L^{q(x)}(\Omega)$ where $\frac{1}{p(x)} + \frac{1}{q(x)} = 1, for\ all\ x \in \Omega$.
Morever if $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$ then $uv \in L^{s(x)}(\Omega)$ and we have the generalised Hölder's type inequality

$$\|uv\|_{s(.)} \le 2\|u\|_{p(.)}.\|v\|_{q(.)}, \quad \frac{1}{s(x)} = \frac{1}{p(x)} + \frac{1}{q(x)}.$$

Lemma 1. *If p is a mesurable function on Ω then for any $u \in L^{p(x)}(\Omega)$ we have*

$$\min\left(\|u\|_{p(x)}^{p-}, \|u\|_{p(x)}^{p+}\right) \le \mathcal{P}_{p(.)}(u) \le \max\left(\|u\|_{p(x)}^{p-}, \|u\|_{p(x)}^{p+}\right).$$

For any nonnegative integer k the variable exponent Sobolev space is defined

$$W^{k,p(x)}(\Omega) = \left\{u \in L^{p(x)}(\Omega) : |\alpha| \le K \Longrightarrow D^\alpha u \in L^{p(x)}(\Omega)\right\},$$

endowed with the norm

$$\|u\|_{W^{k,p(x)}} = \sum_{|\alpha|\le k} |D^\alpha u|_{L^{p(x)}(\Omega)}.$$

Then $W^{k,p(x)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_{W^{k,p(x)}}$. In this way $L^{p(x)}(\Omega)$, $W^{k,p(x)}(\Omega)$ and $W_0^{k,p(x)}(\Omega)$ are separable and reflexive Banach spaces.
We shall frequently use the generalized Poincaré's inequality in $W_0^{1,p(x)}(\Omega)$ given by

$$\exists C > 0, \|u\|_{p(x)} \le C\|\nabla_x u\|_{p(x)}, \quad , for\ all\ u \in W_0^{1,p(x)}(\Omega).$$

Definition 1. *We define the function space of our problem and its norm as follows*

$$\mathcal{V}(\Omega) = \left\{u|u \in W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega), |\Delta_x u| \in W_0^{1,p(x)}(\Omega)\right\},$$

with the norm

$$\|u\|_{\mathcal{V}(\Omega)} = \|u\|_{W_0^{1,p(x)}(\Omega)} + \|u\|_{W^{2,p(x)}(\Omega)} + \|\Delta_x u\|_{W_0^{1,p(x)}(\Omega)}.$$

Lemma 2. *[18, Theorem 4.4] Let Ω is a bounded domain with Lipschitz boundary. In the space $X = W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$ the norm $\|.\|_X$ and $\|\Delta_x.\|_{L^{p(x)}(\Omega)}$ are equivalent norms.*

Lemma 3. *[1, Theorem 5.4] Let Ω be a domain in $\mathbb{R}^n$ that has the cone property then for $n > p$ and $p \le q \le \frac{np}{n-p}$ there exist the following imbeddings*

$$W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \hookrightarrow w_0^{1,q}(\Omega) \hookrightarrow L^q(\Omega). \tag{2.2}$$

Lemma 4. *[19, Lemma 2.1] Assume that $L(t)$ is is a twice continuously differentiable function satisfying*

$$\begin{cases} L''(t) + L'(t) \ge C_0(t+\theta)^\beta L^{1+\alpha}(t), & t > 0 \\ L(0) > 0, L'(0) \ge 0, \end{cases}$$

where $C_0, \theta > 0$, $-1 < \beta \le 0$, $\alpha > 0$ are constants. Then $L(t)$ blows up in finite time.

3. Blow up result

Theorem 1. *Let u be an energy weak solution to problem* (1.1). *Suppose that*

$$2 \le m \le r \quad and \quad 2 \le p(x) \le \frac{2n}{n-2}.$$

Assume further that

$$E(0) = \frac{1}{2}\int_\Omega |u_1|^2 dx + \int_\Omega \frac{1}{p(x)}|\Delta_x u_0|^{p(x)} dx - \frac{b}{r}\int_\Omega |u_0|^r dx \le 0,$$

and

$$\int_\Omega u_0 u_1 dx \ge 0, \tag{3.1}$$

then the solution u blows up on the finite interval $(0, t_{max})$.

Proof. Multiplying Eq (1.1) by $\partial_t u$, and integration by parts over Ω, one has

$$\partial_t \int_\Omega \frac{1}{2}|\partial_t u|^2 dx - \int_\Omega div(\Delta_x \partial_t u \nabla_x u)|\Delta_x u|^{p(x)-2} dx$$
$$+\partial_t \int_\Omega \frac{1}{p(x)}|\Delta_x u|^{p(x)} dx + \mu \int_\Omega |\partial_t u|^m dx = \partial_t \int_\Omega \frac{b}{r}|u|^r dx.$$

So the corresponding energy of solution to (1.1) is defined by

$$E(t) = \frac{1}{2}\int_\Omega |\partial_t u|^2 dx + \int_\Omega \frac{1}{p(x)}|\Delta_x u|^{p(x)} dx - \frac{b}{r}\int_\Omega |u|^r dx. \tag{3.2}$$

In addition

$$\partial_t E(t) = \int_\Omega div(\Delta_x \partial_t u \nabla_x u)|\Delta_x u|^{p(x)-2} dx - \mu \int_\Omega |\partial_t u|^m dx. \tag{3.3}$$

Which gives in turn the following energy identity

$$E(t) + \mu \int_0^t \int_\Omega |\partial_t u|^m dxds = E(0) + \int_0^t \int_\Omega div(\Delta_x \partial_t u \nabla_x u)|\Delta_x u|^{p(x)-2} dxds. \tag{3.4}$$

We define the sets

$$\Omega_- = \{x \in \Omega : |\Delta_x u| < 1\},$$

and

$$\Omega_+ = \{x \in \Omega : |\Delta_x u| \ge 1\}.$$

So by applying Hölder and Young inequality we arrive at

$$\left|\int_\Omega div(\Delta_x \partial_t u \nabla_x u)|\Delta_x u|^{p(x)-2} dx\right|$$
$$= \left|\int_\Omega \nabla_x(\Delta_x \partial_t u)\nabla_x u|\Delta_x u|^{p(x)-2} dx + \int_\Omega \Delta_x \partial_t u \Delta_x u|\Delta_x u|^{p(x)-2} dx\right|$$
$$\le \|\nabla_x(\Delta_x \partial_t u)\|_2 . \|\nabla_x u\|_{\frac{2p^-}{4-p^-}} . \|\Delta_x u\|_{p^-}^{p^- -2}$$
$$+ \|\nabla_x(\Delta_x \partial_t u)\|_2 . \|\nabla_x u\|_{\frac{2p^+}{4-p^+}} . \|\Delta_x u\|_{p^+}^{p^+ -2}$$
$$+ \frac{1}{p^-}\|\Delta_x \partial_t u\|_{p^-}^{p^-} + \frac{p^- - 1}{p^-}\|\Delta_x u\|_{p^-}^{p^-} + \frac{1}{p^+}\|\Delta_x \partial_t u\|_{p^+}^{p^+} + \frac{p^+ - 1}{p^-}\|\Delta_x u\|_{p^+}^{p^+}.$$

Clearly since $2 \leq p^- \leq p(x) \leq p^+ \leq \frac{2n}{n-2}$ then by exploiting lemma 3, we have

$$\begin{aligned}
&\left|\int_\Omega div(\Delta_x \partial_t u \nabla_x u)|\Delta_x u|^{p(x)-2} dx\right| \\
&\leq C_0 \|\nabla_x(\Delta_x \partial_t u)\|_{p^-} . \|\Delta_x u\|_{p^-}^{p^- -1} \\
&+ C_1 \|\nabla_x(\Delta_x \partial_t u)\|_{p^+} . \|\Delta_x u\|_{p^+}^{p^+ -1} \\
&+ \frac{1}{p^-}\|\Delta_x \partial_t u\|_{p^-}^{p^-} + \frac{p^- - 1}{p^-}\|\Delta_x u\|_{p^-}^{p^-} \\
&+ \frac{1}{p^+}\|\Delta_x \partial_t u\|_{p^+}^{p^+} + \frac{p^+ - 1}{p^-}\|\Delta_x u\|_{p^+}^{p^+}.
\end{aligned}$$

Because $\partial_t u$ is regular and by Young inequality we obtain

$$\begin{aligned}
\left|\int_\Omega div(\Delta_x \partial_t u \nabla_x u)|\Delta_x u|^{p(x)-2} dx\right| &\leq k_0 \left(\|\nabla_x(\Delta_x \partial_t u)\|_{p^-}^{p^-} + \|\nabla_x(\Delta_x u)\|_{p^-}^{p^-}\right) \\
&+ k_1 \left(\|\nabla_x(\Delta_x \partial_t u)\|_{p^+}^{p^+} + \|\nabla_x(\Delta_x u)\|_{p^+}^{p^+}\right).
\end{aligned}$$

At this step we will assume that

$$\sup_{0 \leq t \leq t_{max}} \left(\|\nabla_x(\Delta_x \partial_t u)\|_{p^-}^{p^-} + \|\nabla_x(\Delta_x u)\|_{p^-}^{p^-} + \|\nabla_x(\Delta_x \partial_t u)\|_{p^+}^{p^+} + \|\nabla_x(\Delta_x u)\|_{p^+}^{p^+}\right) \leq \frac{|E(0)|}{k t_{max}}, \tag{3.5}$$

where $k = \max(k_0, k_1)$. We notice that estimate (3.5) will be important to prove the blow-up result. Therfore

$$\left|\int_0^t \int_\Omega div(\Delta_x \partial_t u \nabla_x u)|\Delta_x u|^{p(x)-2} dx ds\right| \leq |E(0)|, \quad 0 \leq t \leq t_{max}.$$

Consequently by virtue of (3.4) we derive the following estimate for the energy functional

$$E(t) + \mu \int_0^t \int_\Omega |\partial_t u|^m dx ds \leq E(0) + |E(0)|. \tag{3.6}$$

Suppose that $E(0) \leq 0$ then it follows from (3.6) that $E(t) \leq 0$. Define the auxiliary function $L(t)$ by the following formula

$$L(t) = \frac{1}{2}\int_\Omega |u(x,t)|^2 dx + N \int_0^t H(s) ds + N(t + t_{max}), \tag{3.7}$$

where $N > 0$ is to be specified later and $H(t)$ is given by

$$H(t) = \alpha |E(0)| t - E(t), \quad \theta \geq \frac{1}{k t_{max}}. \tag{3.8}$$

We differentiate (3.8) and use the Eq (3.4) to arrive at

$$\partial_t H(t) = \mu \int_0^t \|\partial_t u\|_m^m - \int_0^t \int_\Omega div(\Delta_x \partial_t u \nabla_x u)|\Delta_x u|^{p(x)-2} dx - (1 + \theta t) E(0). \tag{3.9}$$

Therfore

$$\partial_t H(t) \geq \|\partial_t u\|_m^m + \left(\frac{1}{k t_{max}} - \theta\right) E(0). \tag{3.10}$$

From (3.8) we see that H is a nondecreasing function and

$$H(0) = -E(0) > 0.$$

Differentiating (3.7) twice leads to

$$\begin{aligned} L'(t) &= \int_\Omega u\partial_t u dx + NH(t) + N \\ L''(t) &= \int_\Omega u\partial_{tt} u dx + \int_\Omega |\partial_t u|^2 dx + N\partial_t H(t). \end{aligned} \tag{3.11}$$

It's clear from (3.1) and (3.11) that

$$L(0) > 0, \quad \partial_t L(0) > 0.$$

Now, by using Young's inequality we have

$$\int_\Omega |\Delta_x u|^{p(x)-2}|\nabla_x u||\nabla_x(\Delta_x u)|dx \leq C\left(\|\nabla_x(\Delta_x u)\|_{p^-}^{p^-} + \|\nabla_x(\Delta_x u)\|_{p^+}^{p^+}\right).$$

Again Young's inequality yields

$$\int_\Omega u\partial_t u|\partial_t u|^{m-2}dx \leq \frac{\beta^m}{m}\|u\|_m^m + \frac{m-1}{m}\beta^{-m/m-1}\|\partial_t u\|_m^m, \tag{3.12}$$

where β in an arbitrary nonnegative constant to be specified later. By combining (3.3) and (3.5) we get

$$\begin{aligned} \mu\|\partial_t u\|_m^m &= -\partial_t E(t) - \int_\Omega div(\Delta_x \partial_t u \nabla_x u)|\Delta_x u|^{p(x)-2}dx \\ &\leq -\partial_t E(t) - \frac{E(0)}{t_{max}} \\ &\leq \partial_t H(t) + \alpha E(0) + \frac{H(0)}{t_{\max}} \\ &\leq \partial_t H(t) + \frac{H(t)}{t_{\max}}. \end{aligned} \tag{3.13}$$

Inserting (3.13) into (3.12) leads to

$$\int_\Omega u\partial_t u|\partial_t u|^{m-2}dx \leq \frac{\beta^m}{m}\|u\|_m^m + \frac{m-1}{m}\beta^{-m/m-1}\left(\partial_t H(t) + \frac{H(t)}{t_{\max}}\right). \tag{3.14}$$

By virtue of (3.5) we have

$$-\left(\|\nabla_x(\Delta_x u)\|_{L^{p^-}(\Omega)}^{p^-} + \|\nabla_x(\Delta_x u)\|_{L^{p^+}(\Omega)}^{p^+}\right) \geq \frac{E(0)}{kt_{max}} \geq -\frac{H(t)}{kt_{max}}. \tag{3.15}$$

We define the sets

$$\Omega_- = \{x \in \Omega : |u| < 1\},$$

and

$$\Omega_+ = \{x \in \Omega : |u| \geq 1\}.$$

So

$$\int_\Omega |u|^m dx = \int_{\Omega_-} |u|^m dx + \int_{\Omega_+} |u|^m dx \leq \int_{\Omega_-} |u|^2 dx + \int_{\Omega_+} |u|^r dx. \tag{3.16}$$

We first note that

$$\int_\Omega |u|^2 dx \le C_0 \int_\Omega \left(|u|^{\frac{2p^+}{4-p^+}} dx\right)^{\frac{4-p^+}{p^+}} \le C_1 \left(1 + \|\nabla_x \Delta_x u\|^{p^+}_{L^{p^+}(\Omega)}\right).$$

Therfore from (3.15) we have

$$\begin{aligned} \int_\Omega |u|^m dx &\le \Delta_x \left(1 + \|\nabla_x \Delta_x u\|^{p^-}_{p^-} + \|\nabla_x \Delta_x u\|^{p^+}_{p^+} + \|u\|^r_r\right) \\ &\le \Delta_x \left(1 + \frac{H(t)}{kt_{max}} + \|u\|^r_r\right). \end{aligned} \tag{3.17}$$

Consequently

$$\begin{aligned} L''(t) + L'(t) &= \int_\Omega u\Delta_x(div(|\Delta_x u|^{p(x)-2}\nabla_x u))dx - \mu|\partial_t u|^{m-2}\partial_t uu + b|u|^r dx \\ &+ \|\partial_t u\|^2_2 + \int_\Omega u\partial_t u dx + NH(t) + N\partial_t H(t) + N \\ &\ge -C\left(\|\nabla_x(\Delta_x u)\|^{p^-}_{p^-} + \|\nabla_x(\Delta_x u)\|^{p^+}_{p^+}\right) \\ &- \mu\left(\frac{\beta^m}{m}\|u\|^m_m + \frac{m-1}{m}\beta^{-m/m-1}\|\partial_t u\|^m_m\right) + b\|u\|^r_r \\ &+ \|\partial_t u\|^2_2 + \int_\Omega u\partial_t u dx + NH(t) + N\partial_t H(t) + N. \end{aligned} \tag{3.18}$$

Combination of (3.15) and (3.2) leads to

$$\begin{aligned} \int_\Omega u\partial_t u dx &\le \frac{1}{2}\|\partial_t u\|^2_2 + \sigma\left(1 + \|\nabla_x \Delta_x u\|^{p^-}_{p^-}\right) \\ &\le \frac{1}{2}\|\partial_t u\|^2_2 + \sigma\left(1 + \frac{H(t)}{kt_{max}}\right). \end{aligned} \tag{3.19}$$

Substituting (3.14), (3.17) and (3.19) into (3.18) we obtain

$$\begin{aligned} L''(t) + L'(t) &\ge \left(N - \frac{C}{kt_{max}} - \mu\beta^{-m/m-1}\frac{m-1}{mt_{max}} - \mu\Delta_x\frac{\beta^m}{mkt_{max}} - \frac{\sigma}{kt_{max}}\right)H(t) \\ &+ \frac{1}{2}\|\partial_t u\|^2_2 + \left(N - \mu\frac{m-1}{m}\beta^{-m/m-1}\right)H'(t) \\ &+ \left(b - \mu\frac{\beta^m}{m}\Delta_x\right)\|u\|^r_r + N - \mu\Delta_x\frac{\beta^m}{m} - \sigma. \end{aligned} \tag{3.20}$$

Now we pick β so small that

$$b - \mu\frac{\beta^m}{m}\Delta_x > 0. \tag{3.21}$$

Once β is chosen we select N large enough that

$$\begin{aligned} N - \frac{C}{kt_{max}} - \mu\beta^{-m/m-1}\frac{m-1}{mt_{max}} - \mu\Delta_x\frac{\beta^m}{mkt_{max}} - \frac{\sigma}{kt_{max}} &> 0 \\ N - \mu\frac{m-1}{m}\beta^{-m/m-1} &> 0 \\ N - \mu\Delta_x\frac{\beta^m}{m} - \sigma &> 0. \end{aligned} \tag{3.22}$$

Therfore from (3.21) and (3.22) there exists a constant γ such that (3.20) takes the form

$$L''(t)L(t) + L'(t)L(t) \geq \gamma \|u\|^r_{L^r(\Omega)}. \tag{3.23}$$

Now we use Hölder inequality to estimate the term $\|u\|^r_{L^r(\Omega)}$ as follows

$$\begin{aligned} \int_\Omega |u|^2 dx &\leq |\Omega|^{r-2/r} . \|u\|^2_r \\ &\leq (N(t+t_{max}))^{r-2/r} |\Omega|^{r-2/r} . \|u\|^2_r . \end{aligned} \tag{3.24}$$

Hence

$$\|u\|^r_r \geq |\Omega|^{2-r/2} . (N(t+t_{max}))^{2-r/2} . \|u\|^r_2, \tag{3.25}$$

and from the definition of $L(t)$ in (3.7) we have

$$\begin{aligned} (2L(t))^{r/2} &\leq \|u\|^r_2 + \left(N \int_0^t H(s)ds + N(t+t_{max}) \right)^{r/2} \\ &\leq 2^{r-2/2} \left(\|u\|^r_2 + \left(N \int_0^t H(s)ds + N(t+t_{max}) \right)^{r/2} \right). \end{aligned} \tag{3.26}$$

This gives

$$\|u\|^r_2 \geq 2(L(t))^{r/2} - \left(N \int_0^t H(s)ds + N(t+t_{max}) \right)^{r/2} \geq (L(t))^{r/2}. \tag{3.27}$$

Combining (3.23) and (3.27) yields

$$L''(t) + L'(t) \geq \gamma |\Omega|^{2-r/2} (N(t+t_{max}))^{2-r/2} (L(t))^{r/2}. \tag{3.28}$$

We see that the requirements of theorem 1 are satisfied with

$$-1 < \frac{2-r}{2} \leq 0, \quad \alpha = \frac{r-2}{2} > 0, \quad C_0 = \gamma |\Omega|^{2-r/2} N^{2-r/2} > 0. \tag{3.29}$$

Therefore, L blows up in finite time. This completes the proof. □

4. Conclusions

Let us pass to a survey of the results and methods of proving non-existence and blow-up theorems applicable to equations of hyperbolic type. Here it is necessary to clarify what is meant by the term“destruction of the solution”. By this term, we understand the existence of a finite time moment atwhich the solution of the evolutionary problem leaves the smoothness class to which this solutionbelonged on the interval (0, Tmax) (the smoothness class for which the local solvability theoremis formulated and proved). Looking ahead, we note that in all problems for nonlinear equationsconsidered in the literature, the destruction of the solution is accompanied by the inversion of thenorm of the latter to infinity (in the space where we are looking for a solution), however, such behaviorof solution is not at all necessary in the concept of destruction.

Acknowledgments

The researchers would like to thank the Deanship of Scientific Research, Qassim University forfunding the publication of this project.

Conflict of interest

The authors declare there is no conflict of interest.

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