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Aims and Scope

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Contents

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MOMENTS OF ORDER STATISTICS OF THE POISSON-LOMAX

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ABSTRACT

The Poisson-Lomax distribution has been proposed as a useful reliabilitymodel for analyzing lifetime data. For this distribution, some recurrence relations areestablished for the single moments and product moments of order statistics. Usingthese recurrence relations, the means, variances and covariances of all order statisticscan be computed for all sample sizes in a simple and ecient recursive manner.

1. INTRODUCTION

Order statistics arise naturally in many life applications. The use of recurrence relations for the moments of order statistics is quite well known in statistical literature (see for example Arnold et al. [2], Malik et al. [6]). For improved form of these results, Samueland Thomes [7] have reviewed many recurrence relations and identities for the momentsof order statistics arising from several speci c continuous distributions such as normal,Cauchy, logistic, gamma and exponential. Balakrishnan et. al [11] and Balakrishnanet. al [8] studied recurrence relations and identities for moments of order statistics forspecic continuous distributions. Recurrence relations for the expected values of certainfunctions of two order statistics have been considered by Ali and Khan [1] and Khan et.al [5]. The moments of order statistics have some important applications in inferentialmethods. For an elaborate treatment on the theory, methods and applications of orderstatistics, interested readers may refer to Balakrishnan and Rao [9] and [10].The Poisson-Lomax (PL) distribution, proposed recently by Al- Zahrani and Sagor [3],is a useful model for modeling lifetime data. The distribution is a compound distributionof the zero-truncated Poisson and the Lomax distributions. See also, Al-Awadhi andGhitany [12] for a discrete extension of this model.

Definition 1.1. *We say that a random variable X with range of values $(0,\infty)$ has a Poisson-Lomax distribution, and write $X \sim PL(\alpha,\beta,\lambda)$, if the survival function (sf) is*

$$\bar{F}(x;\alpha,\beta,\lambda) = \frac{1-e^{-\lambda(1+\beta x)^{-\alpha}}}{1-e^{-\lambda}}, \quad x>0,\ \alpha,\ \beta,\ \lambda>0. \tag{1.1}$$

The probability density function (pdf) associated with (1.1) is expressed in a closed form and is given by

$$f(x;\alpha,\beta,\lambda) = \frac{\alpha\beta\lambda(1+\beta x)^{-(\alpha+1)}\, e^{-\lambda(1+\beta x)^{-\alpha}}}{1-e^{-\lambda}}, \quad x>0,\ \alpha,\ \beta,\ \lambda>0. \tag{1.2}$$

It is easy to observe that $f(x)$ and $F(x) = 1-\bar{F}(X)$ satisfy the following characterizing relations:

$$f(x) = c_1(1+\beta x)^{-(\alpha+1)}F(x) + c_2(1+\beta x)^{-(\alpha+1)}, \tag{1.3}$$

where $c_1 = \alpha\beta\lambda$ and $c_2 = c_1 e^{-}\lambda/(1-e^{-\lambda})$. The density function given by (1.2) can be interpreted as a compound of the zero-truncated Poisson distribution and the Lomax distribution. Mathematical properties of this distribution can found in Al-Zahrani and Sagor [3]. Here, we will study the distribution of order statistics and establish some recurrence relations for the single and product moments of order statistics for the Poisson-Lomax distribution. These recurrence relations will enable the computation of the means, variances and covariances of all order statistics for all sample sizes in a simple and efficient recursive manner.

2. Distribution of order statistics

Let $X_1, X_2, \cdots, X_n$ be a random sample of size n from the PL distribution in (1.1) and let $X_{1:n}, \cdots, X_{n:n}$ denote the corresponding order statistics. Then, the pdf of $X_{r:n}$, $1 \le r \le n$, is given by (see David and Nagaraja [4] and Arnold et al. [2])

$$f_{r:n}(x) = C_{r,n}[F(x)]^{r-1}[1-F(x)]^{n-r}f(x), \quad 0 < x < \infty, \tag{2.1}$$

where $C_{r,n} = [B(r, n-r+1)]^{-1}$, with $B(a,b)$ being the complete beta function.

Theorem 2.1. *Let $F(x)$ and $f(x)$ be the cdf and pdf of a Poisson-Lomax distribution for a random variable X. The density of the rth order statistic, say $f_{(r)}(x)$ is given by*

$$\begin{aligned} f_{r:n}(x) &= \alpha\beta\lambda C_{r,n} \sum_{i=0}^{r-1}\sum_{j=0}^{n-r+i} \binom{r-1}{i}\binom{n-r+i}{j} \\ &\quad \times \frac{(-1)^{i+j}(1+\beta x)^{-(\alpha+1)}\ e^{-\lambda(j+1)(1+\beta x)^{-\alpha}}}{(1-e^{-\lambda})^{n-r+i+1}}. \end{aligned} \tag{2.2}$$

Proof. First it should be noted that (2.1) can be written as follows:

$$f_{r:n}(x) = C_{r,n}\sum_{i=0}^{r-1}\binom{r-1}{i}(-1)^i[\bar{F}(x)]^{n-r+i}f(x), \tag{2.3}$$

then the proof follows by replacing the sf, $\bar{F}(x)$, and the pdf, $f(x)$, of $X \sim PL(\alpha, \beta, \lambda)$ which are obtained from (1.1) and (1.2), respectively, substituting them into relation (2.3), and expanding the term $(1-e^{-\lambda(1+\beta x)^{-\alpha}})^{n-r+i}$ using the binomial expansion. □

The distributions of the extreme order statistics are always of great interest. Taking $r = 1$ in equation (2.2), yeilds the pdf of the minimum order statistic

$$f_{1:n}(x) = \frac{n\alpha\beta\lambda(1+\beta x)^{-(\alpha+1)}\ e^{-\lambda(1+\beta x)^{-\alpha}}}{(1-e^{-\lambda})^n}\left[1-e^{-\lambda(1+\beta x)^{-\alpha}}\right]^{n-1},$$

and if we take $r = n$ in equation (2.2), then this yeilds the pdf of the maximum order statistic.

The joint pdf of $X_{r:n}$ and $X_{s:n}$ for $1 \le r < s \le n$ is given by (see e.g. Arnold et al. [2])

$$f_{r,s:n}(x,y) = C_{r,s,n}[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}f(x)f(y), \tag{2.4}$$

where

$$C_{r,s,n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}, \quad -\infty < x < y < \infty.$$

Substituting (1.1) and (1.2) into (2.4), one can obtain the joint pdf of the rth and the sth order statistics from the Poisson-Lomax distribution. It is as follows:

$$\begin{aligned} f_{r,s:n}(x,y) &= \frac{C_{r,s,n}(\alpha\beta\lambda)^2}{(1-e^{-\lambda})^n}[xy\beta^2+(x+y)\beta+1]^{-(\alpha+1)}e^{-\lambda\{(1+\beta x)^{-\alpha}+(1+\beta y)^{-\alpha}\}} \\ &\times\left[e^{-\lambda(1+\beta x)^{-\alpha}}-e^{-\lambda}\right]^{r-1}\left[e^{-\lambda(1+\beta y)^{-\alpha}}-e^{-\lambda(1+\beta x)^{-\alpha}}\right]^{s-r-1} \\ &\times\left[1-e^{-\lambda(1+\beta y)^{-\alpha}}\right]^{n-s}. \end{aligned}$$

3. Single moments

In this section we first give a closed form expression which is derived easily for the kth, $k = 1, 2, \cdots,$ moment of the ith order statistic from the Poisson-Lomax distribution. This formula will be useful in the phase of computation of the identity given.

Theorem 3.1. *Let $X_1, X_2, \cdots, X_n$ be a random sample of size n from the Poisson-Lomax distribution, and let $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$ denote the corresponding order statistics. Then the kth moment of the rth order statistic for $k = 1, 2, \cdots,$ denoted by $\mu_{r:n}^{(k)}$, is given as*

$$\begin{aligned} \mu_{r:n}^{(k)} = E[X_{r:n}^k] &= \alpha C_{r:n}\sum_{j=0}^{n-r}\sum_{i=0}^{k}\sum_{l=0}^{r+j}\sum_{m=0}^{\infty}\binom{n-r}{j}\binom{k}{i}\binom{r+j}{l}(-1)^{j+i+l+m+1} \\ &\times\left(\frac{\lambda^{m+1}(r+j-l)^m e^{-\lambda l}}{m!\beta^k(1-e^{-\lambda})^{r+j}(k-i-\alpha-\alpha m)}\right) \\ &+\frac{\alpha e^{-\lambda}C_{r:n}}{1-e^{-\lambda}}\sum_{j=0}^{n-r}\sum_{i=0}^{k}\sum_{l=0}^{r+j-1}\sum_{m=0}^{\infty}\binom{n-r}{j}\binom{k}{i}\binom{r+j-1}{l} \\ &\times(-1)^{j+i+l+m+1}\left(\frac{\lambda^{m+1}(r+j-1-l)^m e^{-\lambda l}}{m!\beta^k(1-e^{-\lambda})^{r+j-1}(k-i-\alpha-\alpha m)}\right). \end{aligned} \tag{3.1}$$

for $k < i + \alpha + \alpha m$.

Proof. We know that

$$\begin{aligned} \mu_{r:n}^{(k)} &= \int_0^\infty x^k f_{r:n}(x)dx \\ &= C_{r:n}\int_0^\infty x^k[F(x)]^{r-1}[1-F(x)]^{n-r}f(x)dx \\ &= C_{r:n}c_1\int_0^\infty x^k(1+\beta x)^{-(\alpha+1)}[F(x)]^r[1-F(x)]^{n-r}dx \\ &+C_{r:n}c_2\int_0^\infty x^k(1+\beta x)^{-(\alpha+1)}[F(x)]^{r-1}[1-F(x)]^{n-r}dx. \end{aligned} \tag{3.2}$$

(3.3) $$\mu_{r:n}^{(k)} = C_{r:n}c_1 I_1 + C_{r:n}c_2 I_2,$$

where I_1 can be worked out as follows:

$$\begin{aligned}
I_1 &= \int_0^\infty x^k(1+\beta x)^{-(\alpha+1)}\left[F(x)\right]^r\left[1-F(x)\right]^{n-r}dx\\
&= \int_0^\infty x^k\left(1+\beta x\right)^{-(\alpha+1)}\left[F(x)\right]^r\sum_{j=0}^{n-r}\binom{n-r}{j}(-1)^j\left[F(x)\right]^j dx\\
&= \sum_{j=0}^{n-r}\binom{n-r}{j}(-1)^j\int_0^\infty x^k(1+\beta x)^{-(\alpha+1)}\left(1-\frac{1-e^{-\lambda(1+\beta x)^{-\alpha}}}{1-e^{-\lambda}}\right)^{r+j}dx\\
&= \sum_{j=0}^{n-r}\sum_{i=0}^{k}\sum_{l=0}^{r+j}\sum_{m=0}^{\infty}\binom{n-r}{j}\binom{k}{i}\binom{r+j}{l}(-1)^{j+i+l+m+1}\\
&\quad\times\left(\frac{\lambda^m(r+j-l)^m e^{-\lambda l}}{m!\beta^k(1-e^{-\lambda})^{r+j}(k-i-\alpha-\alpha m)}\right).
\end{aligned}$$

Similarly,

$$\begin{aligned}
I_2 &= \frac{\alpha e^{-\lambda}C_{r:n}}{1-e^{-\lambda}}\sum_{j=0}^{n-r}\sum_{i=0}^{k}\sum_{l=0}^{r+j-1}\sum_{m=0}^{\infty}\binom{n-r}{j}\binom{k}{i}\binom{r+j-1}{l}(-1)^{j+i+l+m+1}\\
&\quad\times\left(\frac{\lambda^m(r+j-1-l)^m e^{-\lambda l}}{m!\beta^k(1-e^{-\lambda})^{r+j-1}(k-i-\alpha-\alpha m)}\right).
\end{aligned}$$

Substituting I_1 and I_2 into (3.3) yields (3.1). □

Now we derive recurrence relation for single moments.

Theorem 3.2. *For $k > i+\alpha$, and $1 \le r \le n-1$*

$$\begin{aligned}
\mu_{r:n}^{(k)} &= \sum_{i=0}^{k}\sum_{j=0}^{k-i-\alpha}\frac{1}{k-i-\alpha}\binom{k}{i}\binom{k-i-\alpha}{j}(-1)^i\beta^{j-k-1}\\
&\quad\times\left(c_1 r\mu_{r+1:n}^{(j)} - c_1 r\mu_{r:n}^{(j)} - c_2(n-r+1)\mu_{r-1:n}^{(j)} + c_2 n\mu_{r:n-1}^{(j)}\right).
\end{aligned}\tag{3.4}$$

Proof. Again we use equation (3.2)

$$\mu_{r:n}^{(k)} = C_{r:n}c_1 I_1 + C_{r:n}c_2 I_2,$$

where I_1 is as before:

$$I_1 = \int_0^\infty x^k(1+\beta x)^{-(\alpha+1)}\left(F(x)\right)^r\left(1-F(x)\right)^{n-r}dx.$$

Now using integration by parts, we obtain

$$I_1 = -\sum_{i=0}^{k}\sum_{j=0}^{k-i-\alpha}\frac{(-1)^i\beta^{j-k-1}}{k-i-\alpha}\binom{k}{i}\binom{k-i-\alpha}{j}\Big\{r\int_0^\infty x^j[F(x)]^r[1-F(x)]^{n-r}f(x)dx$$
$$-(n-r)\int_0^\infty x^j[F(x)]^r[1-F(x)]^{n-r-1}f(x)dx.\Big\}.$$

Similarly,

$$I_2 = -\sum_{i=0}^{k}\sum_{j=0}^{k-i-\alpha}\frac{(-1)^i\beta^{j-k-1}}{k-i-\alpha}\binom{k}{i}\binom{k-i-\alpha}{j}$$
$$\times\Big\{(r-1)\int_0^\infty x^j[F(x)]^{r-2}[1-F(x)]^{n-r}f(x)dx$$
$$-(n-r)\int_0^\infty x^j[F(x)]^{r-1}[1-F(x)]^{n-r-1}f(x)dx.\Big\}.$$

Substituting I_1 and I_2 into (3.3) yields (3.4). □

4. Recurrence relations for product moments

Theorem 4.1. *For, $k_2 > i+\alpha$ and $s-r \geq 2$*

$$\mu_{r;s:n}^{(k_1,k_2)} = \sum_{i=0}^{k_2}\sum_{j=0}^{k_2-i-\alpha}\frac{\binom{k_2}{i}\binom{k_2-i-\alpha}{j}}{k_2-i-\alpha}(-1)^i\beta^{j-k_2-1}\Big\{(c_1+c_2)n\mu_{r;s:n-1}^{(k_1,j)}$$
$$-(c_1+c_2)n\mu_{r;s-1:n-1}^{(k_1,j)}+c_1(n-s+1)\mu_{r;s:n}^{(k_1,j)}-c_1(n-s+1)\mu_{r;s-1:n}^{(k_1,j)}\Big\}. \tag{4.1}$$

Proof.

$$\mu_{r;s:n}^{(k_1,k_2)} = C_{r:s;n}\int_0^\infty\int_x^\infty x^{k_1}y^{k_2}[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}$$
$$\times[1-F(y)]^{n-s}f(x)f(y)\,dy\,dx,$$

where $C_{r:s;n} = n!/(r-1)!(s-r-1)!(n-s)!$. Let I be as follows:

$$I = \int_0^\infty\int_x^\infty x^{k_1}y^{k_2}[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}$$
$$\times[1-F(y)]^{n-s}f(x)f(y)\,dy\,dx$$
$$= \int_0^\infty x^{k_1}[F(x)]^{r-1}f(x)I_x dx, \tag{4.2}$$

where

$$I_X = \int_x^\infty y^{k_2}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}f(y)\,dy\,.$$

Using (1.3) we have:

$$\begin{aligned} I_X &= c_1\int_x^\infty y^{k_2}(1+\beta y)^{-(\alpha+1)}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}F(y)\,dy \\ &\quad + c_2\int_x^\infty y^{k_2}(1+\beta y)^{-(\alpha+1)}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}dy\,, \end{aligned}$$

or it can be written as:

$$\begin{aligned} I_X &= c_1\int_x^\infty y^{k_2}(1+\beta y)^{-(\alpha+1)}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}dy \\ &\quad - c_1\int_x^\infty y^{k_2}(1+\beta y)^{-(\alpha+1)}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s+1}dy \\ &\quad + c_2\int_x^\infty y^{k_2}(1+\beta y)^{-(\alpha+1)}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}dy\,. \end{aligned}$$

By using integration by parts, we obtain:

$$\begin{aligned} I_X &= \sum_{i=0}^{k_2}\sum_{j=0}^{k_2-i-\alpha}\frac{(-1)^i\beta^{j-k_2-1}}{k_2-i-\alpha}\binom{k_2}{i}\binom{k_2-i-\alpha}{j} \\ &\quad \times\Bigg\{(c_1+c_2)(n-s)\int_x^\infty y^j[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}f(y)\,dy \\ &\quad -(c_1+c_2)(s-r-1)\int_x^\infty y^j[F(y)-F(x)]^{s-r-2}[1-F(y)]^{n-s}f(y)\,dy \\ &\quad +c_1(n-s+1)\int_x^\infty y^j[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}f(y)\,dy \\ &\quad -c_1(s-r-1)\int_x^\infty y^j[F(y)-F(x)]^{s-r-2}[1-F(y)]^{n-s}f(y)\,dy\Bigg\}. \end{aligned}$$

Substituting I_X in (4.2) and then into (4.1) produces the desired result. □

Corollary 4.1.

$$\begin{aligned} \mu_{r;r+1:n}^{(k_1,\,k_2)} &= \sum_{i=0}^{k_2}\sum_{j=0}^{k_2-i-\alpha}\frac{\binom{k_2}{i}\binom{k_2-i-\alpha}{j}}{k_2-i-\alpha}(-1)^i\beta^{j-k_2-1}\Big\{(c_1+c_2)n\,\mu_{r;r+1:n-1}^{(k_1,j)} \\ &\quad -c_1(n-r)\mu_{r,r+1:n}^{(k_1,j)}-(c_1+c_2)n\,\mu_{r;:n-1}^{(k_1+j)}+c_1(n-r)\,\mu_{r:n}^{(k_1+j)}\Big\}. \end{aligned}$$

Proof. The proof follows by substituting $s=r+1$ into equation (4.1), the rest of procedure is similar to that of Theorem 4.1. □

5. Conclusion

In this paper we studied the Poisson-Lomax distribution from the order statistics viewpoint. Also we considered the single and product moment of order statistics fromthis distribution. We established some recurrence relations for both single and productmoments of order statistics. Using these recurrence relations, one can easily compute the means, variances and covariances of all order statistics for all sample sizes in a simple andecient recursive manner.

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REMARK ON TRIGONOMETRIC FUNCTIONS

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A B S T R A C T

Noting the derivatives for functions sinz and cosz, we assume the fractionalderivatives for sinz and cosz. Applying the fractional derivatives, we consider generalizedexpansions for functions sinz and cosz. Further, the generalized expansion for f(z) = eizis also discussed.

1. INTRODUCTION

Let $\mathcal{A}(\alpha)$ be the class of functions $f(z)$ of the form

$$f(z) = a_0 z^{\alpha} + a_1 z^{\alpha+1} + a_2 z^{\alpha+2} + \cdots = \sum_{n=0}^{\infty} a_n z^{\alpha+n}$$

for $0 \leq \alpha < 1$ which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. If $\alpha = 0$ in (1.1), then $f(z) \in \mathcal{A}(0)$ becomes

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots = \sum_{n=0}^{\infty} a_n z^n,$$

and that

$$f(z) = f(0) + \frac{f'(0)}{1!} z + \frac{f''(0)}{2!} z^2 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n.$$

This is Taylor expansion for $f(z) \in \mathcal{A}(0)$. Therefore, for $f(z) \in \mathcal{A}(\alpha)$, we need to consider the generalization for Taylor expansion of $f(z)$.

To discuss Taylor expansion for $f(z)$ in the class $\mathcal{A}(\alpha)$, we have to introduce the fractional calculus (fractional integrals and fractional derivatives) defined by Owa [1], Owa and Srivastava [2], and Srivastava and Owa [3].

Definition 1.1. *The fractional integral of order α is defined, for an analytic function $f(z)$ in $\mathbb{U}$, by*

$$D_z^{-\alpha} f(z) = \frac{1}{\Gamma(\alpha)} \int_0^z \frac{f(t)}{(z-t)^{1-\alpha}} dt \qquad (\alpha > 0),$$

where the multiplicity of $(z-t)^{\alpha-1}$ is removed by requiring $\log(z-t)$ to be real when $z - t > 0$.

Definition 1.2. *The fractional derivative of order α is defined, for an analytic function $f(z)$ in $\mathbb{U}$, by*

$$D_z^{\alpha} f(z) = \frac{d}{dz}\left(D_z^{\alpha-1} f(z)\right) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz}\left(\int_0^z \frac{f(t)}{(z-t)^{\alpha}} dt\right),$$

where $0 \leq \alpha < 1$ and the multiplicity of $(z-t)^{-\alpha}$ is removed as Definition 1.1 above.

Definition 1.3. *Under the hypotheses of Definition 1.2, the fractional derivative of orde $n+\alpha$ is defined by*

$$D_z^{n+\alpha} f(z) = \frac{d^n}{dz^n}\left(D_z^{\alpha} f(z)\right),$$

where $0 \leq \alpha < 1$ *and* $n \in \mathbb{N}_0 = \{0, 1, 2, \cdots\}$.

By means of Definition 1.2, we have that

$$\begin{aligned} D_z^{\alpha} z^{\alpha+n} &= \frac{d}{dz}\left(D_z^{\alpha-1} z^{\alpha+n}\right) = \frac{d}{dz}\left\{\frac{1}{\Gamma(1-\alpha)}\int_0^z \frac{t^{\alpha+n}}{(z-t)^{\alpha}}dt\right\} \\ &= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dz}\left\{z^{n+1}\int_0^1 \frac{(1-\zeta)^{\alpha+n}}{\zeta^{\alpha}}d\zeta\right\} \qquad (z-t=z\zeta) \\ &= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dz}\left(z^{n+1}B(1-\alpha, \alpha+n+1)\right) = \frac{\Gamma(\alpha+n+1)}{\Gamma(n+1)}z^n, \end{aligned}$$

where $B(x, y)$ is the beta function. Thus, we obtain, for $f(z) \in \mathcal{A}(\alpha)$, that

$$D_z^{\alpha} f(z) = D_z^{\alpha}\left(\sum_{n=0}^{\infty} a_n z^{\alpha+n}\right) = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n+1)}{\Gamma(n+1)} a_n z^n.$$

This gives us that

$$a_0 = \frac{D_z^{\alpha} f(0)}{\Gamma(\alpha+1)}.$$

Since

$$D_z^{\alpha+1} f(0) = \Gamma(\alpha+2)a_1, \qquad a_1 = \frac{D_z^{\alpha+1} f(0)}{\Gamma(\alpha+2)}.$$

Further, we obtain that

$$a_n = \frac{D_z^{\alpha+n} f(0)}{\Gamma(\alpha+n+1)}$$

for $0 \leq \alpha < 1$ and $n \in \mathbb{N}_0$. With the above, we can write that

$$f(z) = \sum_{n=0}^{\infty} \frac{D_z^{\alpha+n} f(0)}{\Gamma(\alpha+n+1)} z^{\alpha+n} \tag{1.1}$$

for $f(z) \in \mathcal{A}(\alpha)$ with $z \neq 0$. Therefore, we use this expansion (1.1) for $f(z) \in \mathcal{A}(\alpha)$.

2. Expansions for trigonometric functions

Let us consider a function $f(z) = \sin z$ for all $z \in \mathbb{U}$. Then it is easy to write that

$$\begin{aligned} f'(z) &= \cos z = \sin\left(z + \frac{\pi}{2}\right), \\ f''(z) &= \cos\left(z + \frac{\pi}{2}\right) = \sin(z + \pi), \end{aligned}$$

and that

$$f^{(n)}(z) = \sin\left(z + \frac{n}{2}\pi\right) \qquad (n \in \mathbb{N}_0). \tag{2.1}$$

With (2.1), we may assume that

$$D_z^{\alpha} f(z) = \sin\left(z + \frac{\alpha}{2}\pi\right) \qquad (0 \leq \alpha < 1) \tag{2.2}$$

and

$$D_z^{\alpha+n} f(z) = \sin\left(z + \frac{\alpha+n}{2}\pi\right) \tag{2.3}$$

for $0 \leq \alpha < 1$ and $n \in \mathbb{N}_0$.

Remark 2.1. *Using the formula* (2.2), *we have*

$$f^{(n)}(z) = D_z^{n-\alpha}\left(D_z^{\alpha} f(z)\right) = D_z^{n-\alpha}\left(\sin\left(z + \frac{\alpha}{2}\pi\right)\right) = \sin\left(z + \frac{n}{2}\pi\right)$$

for $0 \leq \alpha < 1$ *and* $n \in \mathbb{N}_0$.

Now, we derive

Theorem 2.1. *If the equation* (2.2) *is true for* $f(z) = \sin z$, *then*

$$\sin z = \sum_{n=0}^{\infty} \frac{\sin\left(\frac{\alpha+n}{2}\pi\right)}{\Gamma(\alpha+n+1)} z^{\alpha+n} \qquad (z \in \mathbb{U} - \{0\}) \tag{2.4}$$

where $0 \leq \alpha < 1$.

Proof. By (2.2), we see that

$$D_z^{\alpha} f(0) = \sin\left(\frac{\alpha}{2}\pi\right) \qquad (0 \leq \alpha < 1)$$

and by (2.3), we have that

$$D_z^{\alpha+n} f(0) = \sin\left(\frac{\alpha+n}{2}\pi\right) \qquad (0 \leq \alpha < 1, n \in \mathbb{N}_0).$$

This shows us (2.4) with (1.1). [

Corollary 2.1. *If the equation* (2.2) *is true for* $f(z) = \sin z$ *with* $\alpha = \frac{1}{2}$, *then*

$$\sin z = \sum_{n=0}^{\infty} \frac{\sin\left(\frac{2n+1}{4}\pi\right)}{\Gamma\left(n+\frac{3}{2}\right)} z^{n+\frac{1}{2}}$$

$$= \frac{\sqrt{2}\sqrt{z}}{\sqrt{\pi}}\left(1 + \frac{2}{3}z - \frac{2^2}{3\cdot 5}z^2 - \frac{2^3}{3\cdot 5\cdot 7}z^3 + \frac{2^4}{3\cdot 5\cdot 7\cdot 9}z^4 + \frac{2^5}{3\cdot 5\cdot 7\cdot 9\cdot 11}z^5 - \cdots\right)$$

for $z \in \mathbb{U} - \{0\}$.

Next, we try to consider for $f(z) = \cos z$ $(z \in \mathbb{U})$. It is clear that

$$f'(z) = -\sin z = \cos\left(z + \frac{\pi}{2}\right),$$

$$f''(z) = -\sin\left(z + \frac{\pi}{2}\right) = \cos(z + \pi),$$

and that

$$f^{(n)}(z) = \cos\left(z + \frac{n}{2}\pi\right) \qquad (n \in \mathbb{N}_0). \tag{2.5}$$

With the above, we can assume that

$$D_z^{\alpha} f(z) = \cos\left(z + \frac{\alpha}{2}\pi\right) \tag{2.6}$$

and

$$D_z^{\alpha+n} f(z) = \cos\left(z + \frac{\alpha+n}{2}\pi\right) \tag{2.7}$$

for $0 \leq \alpha < 1$ and $n \in \mathbb{N}_0$.

Remark 2.2. *With the formula (2.6), we see that*

$$f^{(n)}(z) = D_z^{n-\alpha}\left(D_z^{\alpha} f(z)\right) = D_z^{n-\alpha}\left(\cos\left(z + \frac{\alpha}{2}\pi\right)\right) = \cos\left(z + \frac{n}{2}\pi\right)$$

for $0 \leq \alpha < 1$ *and* $n \in \mathbb{N}_0$.

For a function $f(z) = \cos z$, we have

Theorem 2.2. *If the equation (2.6) is true for* $f(z) = \cos z$, *then*

$$\cos z = \sum_{n=0}^{\infty} \frac{\cos\left(\frac{\alpha+n}{2}\pi\right)}{\Gamma(\alpha+n+1)} z^{\alpha+n} \qquad (z \in \mathbb{U} - \{0\}) \tag{2.8}$$

with $0 \leq \alpha < 1$.

Proof. Using (2.6), we have

$$D_z^{\alpha} f(0) = \cos\left(\frac{\alpha}{2}\pi\right) \qquad (0 \leq \alpha < 1). \tag{2.9}$$

Also, by (2.7), we see

$$D_z^{\alpha+n} f(0) = \cos\left(\frac{\alpha+n}{2}\pi\right) \qquad (0 \leq \alpha < 1), z \in \mathbb{U} - \{0\}). \tag{2.10}$$

Putting (2.9) and (2.10) in (2.5), we prove the equation (2.8). □

Making $\alpha = \frac{1}{2}$ in Theorem 2.2, we give

Corollary 2.2. *If the equation (2.6) is true for* $\alpha = \frac{1}{2}$, *then*

$$\cos z = \sum_{n=0}^{\infty} \frac{\cos\left(\frac{2n+1}{4}\pi\right)}{\Gamma\left(n+\frac{3}{2}\right)} z^{n+\frac{1}{2}}$$
$$= \frac{\sqrt{2}\sqrt{z}}{\sqrt{\pi}}\left(1 - \frac{2}{3}z - \frac{2^2}{3\cdot5}z^2 + \frac{2^3}{3\cdot5\cdot7}z^3 + \frac{2^4}{3\cdot5\cdot7\cdot9}z^4 - \frac{2^5}{3\cdot5\cdot7\cdot9\cdot11}z^5 - \cdots\right)$$

for $z \in \mathbb{U} - \{0\}$.

Finally, we derive for $f(z) = e^{iz}$.

Theorem 2.3. *If the equation (2.2) and (2.6) are satisfied, then we have*

$$e^{iz} = \sum_{n=0}^{\infty} \frac{\cos\left(\frac{\alpha+n}{2}\pi\right) + i\sin\left(\frac{\alpha+n}{2}\pi\right)}{\Gamma(\alpha+n+1)} z^{\alpha+n} \qquad (z \in \mathbb{N} - \{0\}) \tag{2.11}$$

for $0 \leq \alpha < 1$.

Letting $\alpha = \frac{1}{2}$ in (2.11), we see

Corollary 2.3. *If the equations (2.2) and (2.6) are satisfied for $\alpha = \frac{1}{2}$, then*

$$e^{iz} = \frac{\sqrt{2}\sqrt{z}}{\sqrt{\pi}} \left\{ (1+i) - \frac{2}{3}(1-i)z - \frac{2^2}{3\cdot 5}(1+i)z^2 + \frac{2^3}{3\cdot 5\cdot 7}(1-i)z^3 + \frac{2^4}{3\cdot 5\cdot 7\cdot 9}(1+i)z^4 - \frac{2^5}{3\cdot 5\cdot 7\cdot 9\cdot 11}(1-i)z^5 - \cdots \right)$$

for $z \in \mathbb{U} - \{0\}$.

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ON THE DETERMINATION OF JUMP BY THE DIFFERENTIATED CONJUGATEFOURIER-JACOBI SERIES

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A B S T R A C T

In the present paper we prove a new result on determination of jump discontinuities by the differentiated conjugate Fourier-Jacobi series. Further, we establish Cesàrosummability of the sequence of partial sums of the conjugate Fourier-Chebyshev series, aspecial type of Fourier-Jacobi series which are obtained for $\alpha = \beta = -\frac{1}{2}$.

1. INTRODUCTION

Conjugate Fourier-Jacobi series was introduced by B. Muckenhoupt and E. M. Stein, see [6], when $\alpha = \beta$, and by Zh.-K. Li, see [4], for general α and β. "Conjugacy" is an important concept in classical Fourier analysis which links the study of the more fundamental properties of harmonic functions to that of analytic functions and is used to study the mean convergence of Fourier series, see [11].

Let $P_n^{(\alpha,\beta)}(x)$ be the Jacobi polynomial of degree n and order $(\alpha, \beta), \alpha, \beta > -1$, normalized so that $P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}$. They are orthogonal on the interval $(-1, 1)$ with respect to the measure $d\mu_{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta dx$.

Define $R_n^{(\alpha,\beta)}(x) = \frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(1)}$, and denote by $L_p(\alpha, \beta), (1 \le p < \infty)$ the space of functions $f(x)$ for which $\|f\|_{p(\alpha,\beta)} = \{\int_{-1}^1 |f(x)|^p d\mu_{\alpha,\beta}(x)\}^{\frac{1}{p}}$ is finite.

For functions $f \in L_1(\alpha, \beta)$, its Fourier-Jacobi expansion is

$$f(x) \sim \sum_{n=0}^{\infty} \hat{f}(n)\omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(x),$$

where

$$\hat{f}(n) = \int_{-1}^1 f(y) R_n^{(\alpha,\beta)}(y) d\mu_{\alpha,\beta}(y)\,,$$

are the Fourier coefficients and

$$\omega_n^{(\alpha,\beta)} = \{\int_{-1}^1 [R_n^{(\alpha,\beta)}(y)]^2 d\mu_{\alpha,\beta}(y)\}^{-1} \sim n^{2\alpha+1}\,.$$

With $x = \cos\theta, \theta \in (0,\pi)$, in an equivalent way Fourier-Jacobi expansion is given by

(1.1) $$f(\theta) \sim \sum_{n=0}^{\infty} \hat{f}(n)\omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta),$$

where

$$\hat{f}(n) = \int_0^{\pi} f(\varphi) R_n^{(\alpha,\beta)}(\cos\varphi) d\mu_{\alpha,\beta}(\varphi),$$

(1.2) $$\omega_n^{(\alpha,\beta)} = \{\int_0^{\pi} [R_n^{(\alpha,\beta)}(\cos\varphi)]^2 d\mu_{\alpha,\beta}(\varphi)\}^{-1} \sim n^{2\alpha+1},$$

and correspondingly $d\mu_{\alpha,\beta}(\theta) = 2^{\alpha+\beta+1}\sin^{2\alpha+1}\frac{\theta}{2}\cos^{2\beta+1}\frac{\theta}{2}d\theta$.

To the Fourier-Jacobi series of the form (1.1), its conjugate series is defined by

(1.3) $$\bar{f}(\theta) \sim \frac{1}{2\alpha+2}\sum_{n=1}^{\infty} n\hat{f}(n)\omega_n^{(\alpha,\beta)} R_{n-1}^{(\alpha+1,\beta+1)}(\cos\theta)\sin\theta.$$

Denote by $S_n^{(\alpha,\beta)}(f,x)$ the $n-$th partial sum of (1.1), and by $\bar{S}_n^{(\alpha,\beta)}(f,x)$ the $n-$th partial sum of (1.3), where $x = \cos\theta$. If $\alpha = \beta = -\frac{1}{2}$, the corresponding Fourier-Jacobi series becomes Fourier-Chebyshev series, so by $S_n^{(-\frac{1}{2},-\frac{1}{2})}(f,x)$ we denote the n-th partial sum of the Fourier-Chebyshev series of f.

Also, throughout this paper we use the following general notations: $L[a,b]$ is the space of integrable functions on $[a,b]$ and $C[a,b]$ is the space of continuous function on $[a,b]$ with the uniform norm $\|\cdot\|_{C[a,b]}$. $W[a,b]$ is the space of functions on $[a,b]$ which may have discontinuities only of the first kind and which are normalized by the condition $f(x) = \frac{1}{2}(f(x+) + f(x-))$.

In this paper first we give a review of the results on determination of jump discontinuities for functions of generalized bounded variation by the differentiated Fourier series, and then we prove new results on the determination of jump discontinuities by the differentiated conjugate Fourier-Jacobi series. Further, we prove that the sequence of the conjugate partial sums of Fourier-Chebyshev series is Cesàro summable to 0.

2. Jump of a function and differentiated Fourier series

The knowledge of the precise location of the discontinuity points is essential for many of the methods aiming at obtaining exponential convergence of the Fourier series of a piecewise smooth function, avoiding the well-known Gibbs phenomenon: the oscillatory behavior of the Fourier partial sums of a discontinuous function.

If a function f is integrable on $[-\pi,\pi]$, then it has a Fourier series with respect to the trigonometric system $\{1, \cos nx, \sin nx\}_{n=1}^{\infty}$, and we denote the n-th partial sum of the Fourier series of f by $S_n(x,f)$, i.e.,

$$S_n(x,f) = \frac{a_0(f)}{2} + \sum_{k=1}^{n}(a_k(f)\cos kx + b_k(f)\sin kx),$$

where $a_k(f) = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} f(t)\cos kt\,dt$ and $b_k(f) = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} f(t)\sin kt\,dt$ are the $k-$th Fourier coefficients of the function f. By $\tilde{S}_n(x,f)$ we denote the $n-$th partial sum of the conjugate series, i.e.,

$$\tilde{S}_n(x,f) = \sum_{k=1}^{n}(a_k(f)\sin kx - b_k(f)\cos kx).$$

The identity determining the jumps of a function of bounded variation by means of its differentiated Fourier partial sums has been known for a long time. Let $f(x)$ be a function of bounded variation with period 2π, and $S_n(x,f)$ be the partial sum of order n of its Fourier series. By the classical theorem of Fejer [11] the identity

$$\lim_{n\to\infty}\frac{S_n'(x,f)}{n} = \frac{1}{\pi}(f(x+0)-f(x-0)) \tag{2.1}$$

holds at any point x.

Obviously, Fejér's identity (2.1) is a statement about Cesàro summability of the sequence $\{kb_k\cos kx - ka_k\sin kx\}$, $a_k = a_k(f)$ and $b_k = b_k(f)$ being the k-th cosine and sine coefficient, respectively. As it is well-known, a sequence s_n is Cesàro or $(C,1)$ summable to s if the sequence σ_n of its arithmetical means converges to s, i.e. $\sigma_n = \dfrac{s_0+s_1+\ldots+s_n}{n+1} \to s, n\to\infty$.

Analogously, the sequence s_n is $(C,\alpha), \alpha > -1$, summable to s, if the sequence

$$\sigma_n^{(\alpha)} = \frac{1}{\binom{n+\alpha}{n}}\sum_{k=0}^{n}\binom{n-k+\alpha-1}{n-k}s_k,$$

converges to s.

The concept of higher variation was firstly introduced by N. Wiener, see [10].

A function f is said to be of bounded p-variation, $p\geq 1$, on the segment $[a,b]$ and to belong to the class $\mathcal{V}_p[a,b]$ if

$$V_{a\ p}^{b}(f) = \sup_{\Pi_{a,b}}\left\{\sum_i |f(x_i)-f(x_{i-1})|^p\right\}^{\frac{1}{p}} < \infty,$$

where $\Pi_{a,b} = \{a = x_0 < x_1 < ... < x_n = b\}$ is an arbitrary partition of the segment $[a,b]$. $V_{a\ p}^{b}(f)$ is the p-variation of f on $[a,b]$.

B. I. Golubov, see [2], has shown that identity (2.1) is valid for classes $\mathcal{V}_p$.

Theorem 2.1. *Let $f(x)\in\mathcal{V}_p$, $(1\leq p<\infty)$ and $r\in\mathbb{N}_0$. Then for any point x one has the equation*

$$\lim_{n\to\infty}\frac{S_n^{(2r+1)}(x,f)}{n^{2r+1}} = \frac{(-1)^r}{(2r+1)\pi}(f(x+0)-f(x-0)).$$

Another type of generalization of the class BV on everywhere convergence of Fourier series, for every change of variable, was introduced by D. Waterman in [9].

Let $\Lambda = \{\lambda_n\}$ be a nondecreasing sequence of positive numbers such that $\sum\frac{1}{\lambda_n}$ diverges and $\{I_n\}$ be a sequence of nonoverlapping segments $I_n = [a_n,b_n]\subset[a,b]$. A function f is said to be of Λ-bounded variation on $I=[a,b]$ $(f\in\Lambda BV)$ if

$$\sum\frac{|f(b_n)-f(a_n)|}{\lambda_n} < \infty$$

for every choice of $\{I_n\}$. The supremum of these sums is called the Λ-variation of f on I. In the case $\Lambda = \{n\}$, one speaks of harmonic bounded variation (HBV).

The class HBV contains all Wiener classes. M. Avdispahić has shown in [1] that HBV is the limiting case for validity of the identity (2.1).

G. Kvernadze in [3] generalized Theorem 2.1 for ΛBV classes:

Theorem 2.2. *Let $r \in \mathbb{Z}_+$ and suppose ΛBV is the class of functions of Λ-bounded variation determined by the sequence $\Lambda = (\lambda_k)_{k=1}^{\infty}$. Then*

(a) *the identity*

$$\lim_{n\to\infty} \frac{((S_n(g;\theta))^{(2r+1)}}{n^{2r+1}} = \frac{(-1)^r}{(2r+1)\pi}(g(\theta+) - g(\theta-)).$$

is valid for every $g \in \Lambda BV$ and each fixed $\theta \in [-\pi, \pi]$ if and only if $\Lambda BV \subseteq HBV$.

(b) *there is no way to determine the jump at the point $\theta \in [-\pi, \pi]$ of an arbitrary function $g \in \Lambda BV$ by means of the sequence $((S_n(g;\theta))^{(2r)}, n \in \mathbb{N}_0$.*

Here we also note the result from [3] for the conjugate Fourier series:

Theorem 2.3. *Let $r \in \mathbb{N}$ and suppose ΛBV is the class of functions of Λ-bounded variation determined by the sequence $\Lambda = (\lambda_k)_{k=1}^{\infty}$. Then*

(a) *the identity*

$$\lim_{n\to\infty} \frac{(\widetilde{S}_n(g;\theta))^{(2r)}}{n^{2r}} = \frac{(-1)^{(r+1)}}{2r\pi}(g(\theta+) - g(\theta-)).$$

is valid for every $g \in \Lambda BV$ and each fixed $\theta \in [-\pi, \pi]$ if and only if $\Lambda BV \subseteq HBV$.

(b) *there is no way to determine the jump at the point $\theta \in [-\pi, \pi]$ of an arbitrary function $g \in \Lambda BV$ by means of the sequence $((\widetilde{S}_n(g;\theta))^{(2r+1)}, n \in \mathbb{N}$.*

3. Main Results

Theorem 3.1. *Let $r \in \mathbb{N}$ and suppose ΛBV is the class of functions of Λ-bounded variation determined by the sequence $\Lambda = (\lambda_k)_{k=1}^{\infty}$, and $\alpha \geq -\frac{1}{2}$,*
$\beta \geq -\frac{1}{2}$. Then the identity

$$\lim_{n\to\infty} \frac{[\widetilde{S}_n^{(\alpha,\beta)}(f,x)]^{(2r)}}{n^{2r}} = \frac{(-1)^{(r+1)}}{2r\pi}(1-x^2)^{-r-\frac{1}{2}}[f(x+0) - f(x-0)],$$

is valid for every $f \in \Lambda BV$ and each $x \in (-1,1)$, where $\widetilde{S}_n^{(\alpha,\beta)}(f,x)$ is the n-th partial sum of the conjugate Fourier-Jacobi series, if and only if $\Lambda BV \subseteq HBV$.

Proof. Differentiating an obvious identity, see [8]

$$S_n^{(-\frac{1}{2},-\frac{1}{2})}(f;x) = S_n(g,\theta),$$

where $x = \cos\theta$, $g(\theta) = f(\cos\theta)$ one has

$$\left(S_n^{(-1/2,-1/2)}(f,x)\right)' = S_n'(g,\theta)\cdot\frac{-1}{\sqrt{1-x^2}}.$$

Continuing the differentiation of the last identity with respect to x ($x = \cos\theta$), we obtain by induction the following representation ($r \in \mathbb{N}$) :

$$[S_n^{(-\frac{1}{2},-\frac{1}{2})}(f;x)]^{(2r+1)} =$$

(3.1)
$$= (1-x^2)^{-r-\frac{1}{2}}(S_n(g;\theta))^{(2r+1)} - \sum_{i=1}^{2r} d_i(x)(S_n(g;\theta))^{(i)}$$

for $\theta \in [0,\pi]$, where d_i, $i = 1,2,\dots,2r$, are infinitely differentiable functions on $(-1,1)$. In addition,

(3.2)
$$\|S_n(g;\cdot))^{(i)}\|_{C[-\pi,\pi]} = o(n^{2r+1})$$

for $i = 1,2,\dots,2r$, $r \in \mathbb{N}$, since $g \in W \subset L$, see [3].

By Theorem 4.1 in [7] we have for $\alpha = \beta = -\frac{1}{2}$

$$\lim_{n\to\infty}[\frac{-1}{n}\left(S_n^{(-1/2,-1/2)}(f,x)\right)' - \tilde{S}_n^{(-1/2,-1/2)}(f,x)] = 0$$

thus taking that into account, dividing (3.1) by n^{2r+1} and letting $n\to\infty$ we get

$$\lim_{n\to\infty}\frac{[\widetilde{S}_n^{(-\frac{1}{2},-\frac{1}{2})}(f,x)]^{(2r)}}{n^{2r}} = \lim_{n\to\infty}\frac{1}{n^{2r+1}}[(1-x^2)^{-r-\frac{1}{2}}(S_n(g;\theta))^{(2r+1)} - \sum_{i=1}^{2r} d_i(x)(S_n(g;\theta))^{(i)}]$$

Using the well-known relation $\widetilde{S}_n(g,\theta) = \frac{-1}{n}S_n'(g,\theta)$, we have

$$\lim_{n\to\infty}\frac{[\widetilde{S}_n^{(-\frac{1}{2},-\frac{1}{2})}(f,x)]^{(2r)}}{n^{2r}} = \lim_{n\to\infty}[\frac{-1}{n^{2r}}(1-x^2)^{-r-\frac{1}{2}}(\widetilde{S}_n(g;\theta))^{(2r)} - \frac{1}{n^{2r}}\sum_{i=1}^{2r} d_i(x)(S_n(g;\theta))^{(i)}].$$

By Theorem 2.3 and (3.2) we have further

$$\lim_{n\to\infty}\frac{[\widetilde{S}_n^{(-\frac{1}{2},-\frac{1}{2})}(f,x)]^{(2r)}}{n^{2r}} = -(1-x^2)^{-r-\frac{1}{2}}\frac{(-1)^{(r+1)}}{2r\pi}(g(\theta+) - g(\theta-)).$$

Taking into account that $f(x\pm) = g(\theta\mp)$, $\theta \in [0,\pi]$, we get

$$\lim_{n\to\infty}\frac{[\widetilde{S}_n^{(-\frac{1}{2},-\frac{1}{2})}(f,x)]^{(2r)}}{n^{2r}} = (1-x^2)^{-r-\frac{1}{2}}\frac{(-1)^{(r+1)}}{2r\pi}[f(x+0) - f(x-0)].$$

Finally, using the equiconvergence formula

$$\|\tilde{S}_n^{(\alpha,\beta)}(f,x) - \tilde{S}_n^{(-1/2,-1/2)}(f,x)\|_{C[\Delta(\nu,\varepsilon)]} = o(1),$$

where $\alpha \geq -\frac{1}{2}$ and $\beta \geq -\frac{1}{2}$, proved in [7] (for an arbitrary function $f \in HBV$ and a fixed $\varepsilon \in (0, \frac{x_{\nu+1}-x_\nu}{2})$, $\nu = 0,1,2,\dots,M$, where it is assumed that $x_0 = -1$, $x_{M+1} = 1$ and $\Delta(\nu;\varepsilon) = [x_\nu + \varepsilon; x_{\nu+1} - \varepsilon]$,) we prove the result. □

For $\alpha = \beta = -\frac{1}{2}$ the corresponding Fourier-Jacobi series becomes Fourier-Chebyshev series, so by $\tilde{S}_n^{(-\frac{1}{2},-\frac{1}{2})}(f,x)$ we denote the n-th partial sum of the conjugate Fourier-Chebyshev series of f. Further, we prove that the sequence of the conjugate partial sums of Fourier-Chebyshev series is Cesàro summable to 0.

Theorem 3.2.

$$\lim_{n\to\infty} \frac{\tilde{S}_1^{(-1/2,-1/2)}(f,x) + \tilde{S}_2^{(-1/2,-1/2)}(f,x) + \ldots + \tilde{S}_{n-1}^{(-1/2,-1/2)}(f,x)}{n} = 0,$$

for every $f \in L_1(-1/2,-1/2)$ and each $-1 < x < 1$.

Proof. According to (1.3)

$$\tilde{S}_n^{(-\frac{1}{2},-\frac{1}{2})}(f,x) = \sum_{k=1}^{n} k \cdot \hat{f}(k)\omega_k^{(-\frac{1}{2},-\frac{1}{2})} \cdot R_{k-1}^{(\frac{1}{2},\frac{1}{2})}(\cos\theta)\sin\theta.$$

The sum

$$\tilde{S}_1^{(-1/2,-1/2)}(f,x) + \tilde{S}_2^{(-1/2,-1/2)}(f,x) + \ldots + \tilde{S}_{n-1}^{(-1/2,-1/2)}(f,x)$$

can be written as

$$1(n-1)a_1 + 2(n-2)a_2 + \ldots + (n-1)1a_{n-1},$$

where $a_i = \hat{f}(i)\omega_i^{(-\frac{1}{2},-\frac{1}{2})} R_{i-1}^{(\frac{1}{2},\frac{1}{2})}(\cos\theta)\sin\theta$. First we will use the Stolz-Cesàro theorem, so

$$\lim_{n\to\infty} \frac{1(n-1)a_1 + 2(n-2)a_2 + \ldots + (n-1)\cdot 1 \cdot a_{n-1}}{n} = \lim_{n\to\infty} na_n.$$

In order to prove the equiconvergence we use (1.2), the approximation [8, Theorem 8.21.8]

$$\begin{aligned}
P_n^{(\alpha,\beta)}(\cos\theta) &= n^{-1/2}k(\theta)cos(N\theta+\gamma) + O(n^{-3/2}),\\
k(\theta) &= \pi^{-\frac{1}{2}}(sin\frac{\theta}{2})^{-\alpha-\frac{1}{2}}(cos\frac{\theta}{2})^{\beta-\frac{1}{2}},\\
N &= n + \frac{\alpha+\beta+1}{2},\\
\gamma &= -(\alpha+\frac{1}{2})\frac{\pi}{2}, 0 < \theta < \pi,
\end{aligned}$$

and [5, Lemma 2.3.]

$$\lim_{n\to\infty} n^{\alpha+\frac{1}{2}} \int_{-1}^{1} f(y) R_n^{(\alpha,\beta)}(y) d\mu_{\alpha,\beta}(y) = 0,$$

for $\alpha, \beta > -1$, $f \in L_1(\min(\alpha, \alpha/2 - 1/4), \min(\beta, \beta/2 - 1/4))$, which is a direct generalization of the Riemann-Lebesgue theorem. Finally we get

$$\begin{aligned}
\lim_{n\to\infty} na_n &= \lim_{n\to\infty} n\hat{f}(n)\omega_n^{(-\frac{1}{2},-\frac{1}{2})} R_{n-1}^{(\frac{1}{2},\frac{1}{2})}(\cos\theta)\sin\theta\\
&= \lim_{n\to\infty} n\hat{f}(n)\omega_n^{(-\frac{1}{2},-\frac{1}{2})} \frac{P_{n-1}^{(\frac{1}{2},\frac{1}{2})}(\cos\theta)}{P_{n-1}^{(\frac{1}{2},\frac{1}{2})}(1)} \sin\theta\\
&= 0,
\end{aligned}$$

as $P_{n-1}^{(\frac{1}{2},\frac{1}{2})}(1) \sim (n-1)^{1/2}$.

□

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MAJORIZATION OF BEREZIN TRANSFORM

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ABSTRACT

In this paper, we majorize the Berezin transform of positive invertible operators defined from the Bergman space L2a(D) into itself. We also present sufficient conditions onbounded operators S, T L(L2a(D)) such that ρ(|S|) = ρ(T) in terms of the Schattennorm of these operators. Here ρ(T) is the Berezin transform of T. Further, given T L(L2a(D)), we find conditions on the existence of a projection operator E L(L2a(D))such that ρ(T E) = 0.

1. INTRODUCTION

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and let $dA(z) = \frac{1}{\pi}dxdy$ denote the normalized Lebesgue area measure on $\mathbb{D}$ in the complex plane $\mathbb{C}$. For $1 \leq p < \infty$ and $f : \mathbb{D} \longrightarrow \mathbb{C}$ Lebesgue measurable let $\|f\|_p = \left(\int_{\mathbb{D}} |f|^p dA(z)\right)^{1/p}$. The Bergman space $L_a^p(\mathbb{D})$ is the Banach space of analytic functions $f : \mathbb{D} \longrightarrow \mathbb{C}$ such that $\|f\|_p < \infty$. The Bergman space $L_a^2(\mathbb{D})$ is a Hilbert space; it is a closed subspace [3] of the Hilbert space $L^2(\mathbb{D}, dA)$ with the inner product given by $\langle f, g\rangle = \int_{\mathbb{D}} f(z)\overline{g(z)}dA(z),\ f, g \in L^2(\mathbb{D}, dA)$. Let P denote the orthogonal projection of $L^2(\mathbb{D}, dA)$ onto $L_a^2(\mathbb{D})$. Let $K(z, \bar{w})$ be the function on $\mathbb{D} \times \mathbb{D}$ defined by $K(z, \bar{w}) = \overline{K_z(w)} = \frac{1}{(1-z\bar{w})^2}$. The function $K(z, \bar{w})$ is called the reproducing kernel of $L_a^2(\mathbb{D})$. For any $n \geq 0, n \in \mathbb{Z}$, let $e_n(z) = \sqrt{n+1}z^n$, then $\{e_n\}$ forms an orthonormal basis for $L_a^2(\mathbb{D})$. Let $k_a(z) = \frac{K(z,\bar{a})}{\sqrt{K(a,\bar{a})}} = \frac{1-|a|^2}{(1-\bar{a}z)^2}$. These functions k_a are called the normalized reproducing kernels of $L_a^2(\mathbb{D})$; it is clear that they are unit vectors in $L_a^2(\mathbb{D})$. Let $L^\infty(\mathbb{D}, dA)$ be the Banach space of all essentially bounded measurable functions f on $\mathbb{D}$ with $\|f\|_\infty = \text{ess sup}\{|f(z)| : z \in \mathbb{D}\}$ and $H^\infty(\mathbb{D})$ be the space of bounded analytic functions on $\mathbb{D}$. Let $\mathcal{L}(H)$ be the space of all bounded linear operators from the separable Hilbert space H into itself and $\mathcal{LC}(H)$ be the space of all compact operators in $\mathcal{L}(H)$. An operator $A \in \mathcal{L}(H)$ is called positive if $\langle Ax, x\rangle \geq 0$ holds for every $x \in H$ in which case we write $A \geq 0$. The absolute value of an operator A is the positive operator $|A|$ defined

as $|A| = (A^*A)^{\frac{1}{2}}$. If H is infinite-dimensional, the map $|\cdot|$ on $\mathcal{L}(H)$ is not Lipschitz continuous. We define $\rho : \mathcal{L}(L_a^2(\mathbb{D})) \longrightarrow L^\infty(\mathbb{D})$ by $\rho(T)(z) = \widetilde{T}(z) = \langle Tk_z, k_z\rangle,\ z \in \mathbb{D}$. A function $g(x, \bar{y})$ on $\mathbb{D} \times \mathbb{D}$ is called of positive type (or positive definite), written $g \gg 0$, if

$$\sum_{j,k=1}^{n} c_j\overline{c_k}g(x_j, \overline{x_k}) \geq 0$$

for any n-tuple of complex numbers $c_1, \ldots, c_n$ and points $x_1, \ldots, x_n \in \mathbb{D}$. We write $g \gg h$ if $g - h \gg 0$. We shall say $\Upsilon \in \mathcal{A}$ if $\Upsilon \in L^\infty(\mathbb{D})$ and is such that

$$\Upsilon(z) = \Theta(z, \bar{z}) \tag{1.1}$$

where $\Theta(x, \bar{y})$ is a function on $\mathbb{D} \times \mathbb{D}$ meromorphic in x and conjugate meromorphic in y and if there exists a constant $c > 0$ such that

$$cK(x, \bar{y}) \gg \Theta(x, \bar{y})K(x, \bar{y}) \gg 0 \text{ for all } x, y \in \mathbb{D}.$$

It is a fact that (see [7], [8]) Θ as in (1.1), if it exists, is uniquely determined by Υ. In this paper, we majorize the Berezin transform of positive invertible operators belonging to $\mathcal{L}(L^2_a(\mathbb{D}))$. The organization of this paper is as follows: In Section 2, we find conditions on positive invertible operators $A, B \in \mathcal{L}(L^2_a(\mathbb{D}))$ such that $\rho(XB^{-1}X) \leq \rho(A)$ where $X \in \mathcal{L}(L^2_a(\mathbb{D}))$ is self-adjoint. In Section 3, we establish that if f is an operator monotone function on $[0, \infty)$ and $A \in \mathcal{L}(L^2_a(\mathbb{D}))$ is positive then $\Theta_{f(EAE)}(x, \bar{y})K(x, \bar{y}) \gg \Theta_{Ef(A)E}(x, \bar{y})K(x, \bar{y})$ for all $x, y \in \mathbb{D}$ and $\rho(f(EAE)) = \rho(Ef(A)E)$ if and only if E and A commute $f(0) = 0$ and f is not a linear function. Section 4 is devoted to Schatten norm and contractions. In this section, we obtain sufficient conditions on Schatten norm of $S, T \in \mathcal{L}(L^2_a(\mathbb{D}))$ such that $\rho(|S|) = \rho(T)$ and $\rho(S) \leq \rho(T)$. Further, we also find conditions on the existence of projection operator $E \in \mathcal{L}(L^2_a(\mathbb{D}))$ such that $\rho(TE) = 0$.

2. On invertible positive operators

In this section, we find conditions on positive invertible operators $A, B \in \mathcal{L}(L^2_a(\mathbb{D}))$ such that $\rho(XB^{-1}X) \leq \rho(A)$ where $X \in \mathcal{L}(L^2_a(\mathbb{D}))$ is self-adjoint. If $S \in \mathcal{L}(L^2_a(\mathbb{D}))$ and S is positive, then let $\Theta_S(x, \bar{y}) = \frac{\langle SK_y, K_x\rangle}{\langle K_y, K_x\rangle}$ for all $x, y \in \mathbb{D}$.

Theorem 2.1. *Let $A, B \in \mathcal{L}(L^2_a(\mathbb{D}))$ are positive and invertible and $X \in \mathcal{L}(L^2_a(\mathbb{D}))$ is self-adjoint. Then*

$$\Theta_A(x, \bar{y})K(x, \bar{y}) \gg \Theta_{XB^{-1}X}(x, \bar{y})K(x, \bar{y}) \tag{2.1}$$

if and only if

$$|\langle XK_y, K_x\rangle|^2 \leq \langle AK_x, K_x\rangle\langle BK_y, K_y\rangle \tag{2.2}$$

for all $x, y \in \mathbb{D}$. In this case $\rho(XB^{-1}X) \leq \rho(A)$.

Proof. Suppose (2.1) holds. Then

$$\langle AK_y, K_x\rangle \geq \langle XB^{-1}XK_y, K_x\rangle$$

for all $x, y \in \mathbb{D}$. The last inequality is valid if and only if

$$\sum_{i,j=1}^{n} c_j\bar{c}_i\langle AK_{x_j}, K_{x_i}\rangle \geq \sum_{i,j=1}^{n} c_j\bar{c}_i\langle XB^{-1}XK_{x_j}, K_{x_i}\rangle$$

where $x_1, x_2, \cdots, x_n \in \mathbb{D}$ and $c_j, j = 1, 2, \cdots, n$ are constants. Thus (2.1) holds if and only if

$$\left\langle A\left(\sum_{j=1}^{n} c_j K_{x_j}\right), \left(\sum_{i=1}^{n} c_i K_{x_i}\right)\right\rangle \geq \left\langle XB^{-1}X\left(\sum_{j=1}^{n} c_j K_{x_j}\right), \left(\sum_{i=1}^{n} c_i K_{x_i}\right)\right\rangle.$$

Since $\left\{\sum_{j=1}^{n} c_j K_{x_j}; x_j \in \mathbb{D}, j = 1, \cdots, n\right\}$ is dense in $L_a^2(\mathbb{D})$, hence (2.1) holds if and only if $\langle Ag, g\rangle \geq \langle XB^{-1}Xg, g\rangle$ for all $g \in L_a^2(\mathbb{D})$. That is, if and only if $A \geq XB^{-1}X$. Now considering the congruence

$$\begin{aligned}\begin{pmatrix} A & X \\ X & B \end{pmatrix} &\sim \begin{pmatrix} I & -XB^{-1} \\ 0 & I \end{pmatrix}\begin{pmatrix} A & X \\ X & B \end{pmatrix}\begin{pmatrix} I & 0 \\ -B^{-1}X & I \end{pmatrix} \\ &= \begin{pmatrix} A - XB^{-1}X & 0 \\ 0 & B \end{pmatrix}\end{aligned}$$

we obtain $A \geq XB^{-1}X$ if and only if $\begin{pmatrix} A & X \\ X & B \end{pmatrix}$ is positive. Thus (2.1) holds if and only if $\begin{pmatrix} A & X \\ X & B \end{pmatrix}$ is positive. Suppose $\begin{pmatrix} A & X \\ X & B \end{pmatrix} \geq 0$ in $\mathcal{L}(L_a^2 \oplus L_a^2)$. Then it follows from [2], that $\left|\left\langle\begin{pmatrix} A & X \\ X & B \end{pmatrix}\begin{pmatrix} K_x \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ K_y \end{pmatrix}\right\rangle\right|^2$
$\leq \left\langle\begin{pmatrix} A & X \\ X & B \end{pmatrix}\begin{pmatrix} K_x \\ 0 \end{pmatrix}, \begin{pmatrix} K_x \\ 0 \end{pmatrix}\right\rangle\left\langle\begin{pmatrix} A & X \\ X & B \end{pmatrix}\begin{pmatrix} 0 \\ K_y \end{pmatrix}, \begin{pmatrix} 0 \\ K_y \end{pmatrix}\right\rangle$ for all $x, y \in \mathbb{D}$. A simplification of these inner products yields

$$|\langle XK_x, K_y\rangle|^2 \leq \langle AK_x, K_x\rangle\langle BK_y, K_y\rangle$$

for all $x, y \in \mathbb{D}$. That is,

$$|\langle XK_y, K_x\rangle|^2 \leq \langle AK_x, K_x\rangle\langle BK_y, K_y\rangle$$

for all $x, y \in \mathbb{D}$. That is, (2.2) holds. Suppose (2.2) holds for all $x, y \in \mathbb{D}$. Let $f = \sum_{j=1}^{n} c_j K_{y_j}$ and $g = \sum_{i=1}^{m} d_i K_{x_i}$ where c_j are constants for $j = 1, 2, \cdots, n$ and d_i are constants, $x_i \in \mathbb{D}$, for $i = 1, 2, \cdots, m$. Then using Heinz inequality [5] we obtain

$$|\langle Xf, g\rangle|^2 \leq \langle |X|f, f\rangle\langle |X|g, g\rangle \tag{2.3}$$

for all $f, g \in L_a^2(\mathbb{D})$. Now it follows from (2.3), that

$$\begin{aligned}\left\langle\begin{pmatrix} A & X \\ X & B \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix}\right\rangle &= \langle Af, f\rangle + \langle Xg, f\rangle + \langle Xf, g\rangle + \langle Bg, g\rangle \\ &= \langle Af, f\rangle + \langle Bg, g\rangle + 2\text{Re}\langle Xf, g\rangle \\ &\geq 2\langle Af, f\rangle^{1/2}\langle Bg, g\rangle^{1/2} + 2\text{Re}\langle Xf, g\rangle \\ &\geq 2|\langle Xf, g\rangle| + 2\text{Re}\langle Xf, g\rangle \\ &\geq 2|\langle Xf, g\rangle| - 2|\langle Xf, g\rangle| = 0\end{aligned}$$

for all $f, g \in L_a^2(\mathbb{D})$. Hence $\begin{pmatrix} A & X \\ X & B \end{pmatrix}$ is positive. From the first part it follows that $A \geq XB^{-1}X$ and

$$\Theta_A(x,\bar{y})K(x,\bar{y}) \gg \Theta_{XB^{-1}X}(x,\bar{y})K(x,\bar{y})$$

for all $x, y \in \mathbb{D}$. The result follows. □

Corollary 2.1. *Let $0 < m \leq A \leq M$ and E is a projection operator from $L_a^2(\mathbb{D})$ onto a closed subspace $\mathcal{M}$. Let $A^{-1}\mid_{\mathcal{M}} = A_1$ and $A\mid_{\mathcal{M}} = A_2$. Then*

$$\Theta_{EA_1}(x,\bar{y})K(x,\bar{y}) \gg \Theta_{(EA_2)^{-1}}(x,\bar{y})K(x,\bar{y}) \tag{2.4}$$

for all $x, y \in \mathbb{D}$. Further $\rho(EA_1) = \rho((EA_2)^{-1})$ if and only if E and A commute.

Proof. The inequality in (2.4) follows from Theorem 2.1. Notice that EA_1 and $(I-E)A^{-1}\mid_{\mathcal{M}^\perp}$ are invertible and

$$(EA_2)^{-1} = EA_1 - EA^{-1}((I-E)A^{-1}\mid_{\mathcal{M}^\perp})^{-1}(I-E)A_1.$$

Now let $(EA_2)^{-1} = EA_1$. Then $(I-E)A_1 = 0$ and this implies $EA^{-1} = A^{-1}E$. Thus $EA = AE$. □

3. Operator monotone function

In this section, we establish that if f is an operator monotone function on $[0,\infty)$ and $A \in \mathcal{L}(L_a^2(\mathbb{D}))$ is positive then

$$\Theta_{f(EAE)}(x,\bar{y})K(x,\bar{y}) \gg \Theta_{Ef(A)E}(x,\bar{y})K(x,\bar{y})$$

for all $x, y \in \mathbb{D}$ and $\rho(f(EAE)) = \rho(Ef(A)E)$ if and only if E and A commute $f(0) = 0$ and f is not a linear function.

Theorem 3.1. *Let f be an operator monotone function on $[0,\infty)$ and assume $f(0) \geq 0$. Let $A \in \mathcal{L}(L_a^2(\mathbb{D}))$ and $A \geq 0$ and E is the projection operator from $L_a^2(\mathbb{D})$ onto a closed subspace $\mathcal{M}$ of $L_a^2(\mathbb{D})$. Then*

$$\Theta_{f(EAE)}(x,\bar{y})K(x,\bar{y}) \gg \Theta_{Ef(A)E}(x,\bar{y})K(x,\bar{y})$$

for all $x, y \in \mathbb{D}$. Further, $\rho(f(EAE)) = \rho(Ef(A)E)$ if and only if E and A commute $f(0) = 0$ and f is not a linear function.

Proof. Since f is operator monotone on $[0,\infty)$, hence f can be represented as

$$f(s) = a + bs + \int_0^\infty \left(\frac{1}{t} - \frac{1}{t+s}\right) d\mu(t)$$

where $a = f(0), b \geq 0$ and μ is a positive Borel measure such that

$$\int_0^\infty \frac{1}{1+t^2} d\mu(t) < \infty.$$

Let E be the projection operator from $L_a^2(\mathbb{D})$ onto the closed subspace $\mathcal{M}$ of $L_a^2(\mathbb{D})$. Then

$$\langle Ef(A)Eg, g\rangle = \langle (a+bA)Eg, Eg\rangle + \int_0^\infty \left(\frac{1}{t}I - (tI+A)^{-1}Eg, Eg\right) d\mu(t)$$

and

$$\langle f(EAE)g,g\rangle = \langle (a+bEAE)g,g\rangle + \int_0^\infty \left\langle \left(\frac{1}{t}I - (tI+EAE)^{-1}\right) g,g\right\rangle d\mu(t)$$
$$= \langle (a+bEAE)g,g\rangle + \int_0^\infty \left\langle \left(\frac{1}{t}E - (E(tI+A)\mid_{\mathcal{M}})^{-1}\right) Eg,Eg\right\rangle d\mu(t).$$

By Corollary 2.1,

$$\frac{1}{t}E - (E(tI+A)\mid_{\mathcal{M}})^{-1} \geq \frac{1}{t}E - E(tI+A)^{-1}E$$

for $t>0$ implies that $f(EAE)\geq Ef(A)E$. Thus

$$\Theta_{f(EAE)}(x,\bar{y})K(x,\bar{y}) \gg \Theta_{Ef(A)E}(x,\bar{y})K(x,\bar{y})$$

for all $x,y\in\mathbb{D}$. Now if, $f(EAE)=Ef(A)E$, then for every $g\in L^2_a(\mathbb{D})$, $\langle ag,g\rangle = \langle aEg,Eg\rangle$ and

$$\langle (E(tI+A)\mid_{\mathcal{M}})^{-1}Eh,Eh\rangle = \langle (tI+A)^{-1}Eh,Eh\rangle$$

for almost every $t>0$ with respect to μ. Since $L^2_a(\mathbb{D})$ is separable, we obtain

$$(E(tI+A)\mid_{\mathcal{M}})^{-1} = E(tI+A)^{-1}E$$

for almost every $t>0$. Thus by Corollary 2.1, $E(tI+A)=(tI+A)E$ and hence $EA=AE$ and $f(0)=a=0$. □

4. Schatten norm and contractions

In this section, we obtain sufficient conditions on Schatten norm of $S,T\in\mathcal{L}(L^2_a(\mathbb{D}))$ such that $\rho(|S|)=\rho(T)$ and $\rho(S)\leq\rho(T)$. Further, we also find conditions on the existence of $E\in L^2_a(\mathbb{D})$ such that $\rho(TE)=0$. From [5], it follows that if $T\in\mathcal{L}(H)$, then $|\langle Tx,y\rangle|^2\leq\langle |T|^{2\alpha}x,x\rangle\langle |T^*|^{2(1-\alpha)}y,y\rangle$ for all $x,y\in H$ and for $0\leq\alpha\leq 1$. An operator $T\in\mathcal{LC}(H)$ is said to be in the Schatten p-class $S_p(H)$ $(1\leq p<\infty)$, if trace$(|T|^p)<\infty$. Let S_∞ be the set of all bounded operators from $L^2_a(\mathbb{D})$ into itself. The Schatten p-norm of T is defined by $\|T\|_p = (\text{trace}|T|^p)^{1/p}$. It is well known that if $T\in S_p(H)$ then, $\|T\|_p=\|T^*\|_p=\||T|\|_p$. The class $S_1(H)$ is also called the trace class of H.

$$\|T\|_1 = \text{trace}|T| = \|T\|_{tr} = \sum_{k=1}^\infty |\langle T\phi_k,\phi_k\rangle|$$

where $\{\phi_k\}$ is an orthonormal basis for H. Let x and y be two nonzero vectors in H. Suppose $\langle x,y\rangle=0$. Let $T=x\otimes y+y\otimes x$. Then T is self-adjoint on H. Further, $\|T^2\|_p = 2^{\frac{1}{p}}\|x\|^2\|y\|^2$ and $\|T\|_p = 2^{\frac{1}{p}}\|x\|\|y\|$, where $\|\cdot\|_p$ is the Schatten p-class norm for $p\geq 1$. Thus $\|T^2\|_p\neq\|T\|_p^2$. Notice that $T^2=\|y\|^2x\otimes x+\|x\|^2y\otimes y$, so the square root $|T|$ of the positive operator T^2 is

$$|T| = \|x\|\|y\|\frac{x}{\|x\|}\otimes\frac{x}{\|x\|} + \|x\|\|y\|\frac{y}{\|y\|}\otimes\frac{y}{\|y\|}.$$

Proposition 4.1. *Let T be a rank k normal operator on H with $\{\lambda\}_{j=1}^k$ the k eigenvalues of T repeated according to multiplicity. Then*

$$trace|T^2| \leq (trace|T|)^2 \leq k\,trace|T^2|.$$

Proof. Notice that $\|T\|_{\mathrm{tr}} = \sum_{j=1}^{k} |\lambda_j|$. Since T^2 is also of rank k and normal with the eigenvalues $\{\lambda_j^2\}_{j=1}^{k}$, by functional calculus, $\|T^2\|_{\mathrm{tr}} = \sum_{j=1}^{k} |\lambda_j|^2$. So the first inequality is trivial. The second inequality follows from the Cauchy-Schwarz inequality. □

Proposition 4.2. *Let $S, T \in \mathcal{L}(L^2_a(\mathbb{D}))$. If $tr(ESE) = tr(ETE)$ for every rank-one projection $E \in \mathcal{L}(L^2_a(\mathbb{D}))$, then $\rho(S) = \rho(T)$.*

Proof. For $z \in \mathbb{D}$, let $E = k_z \otimes k_z$ where $k_z \in L^2_a(\mathbb{D})$ is the normalized reproducing kernel. Then E is a rank-one projection and every rank-one projection takes this form. By the assumption, we have

$$\begin{aligned} \langle Sk_z, k_z\rangle &= \mathrm{tr}(Sk_z \otimes k_z) \\ &= \mathrm{tr}(ESE) = \mathrm{tr}(ETE) \\ &= \mathrm{tr}(Tk_z \otimes k_z) = \langle Tk_z, k_z\rangle. \end{aligned}$$

Thus for all $z \in \mathbb{D}$, $\langle Sk_z, k_z\rangle = \langle Tk_z, k_z\rangle$ and $\rho(S) = \rho(T)$. □

Lemma 4.1. *Let $S, T \in S_p$ for some $p \in [1, \infty)$. If $0 \leq S \leq T$ and $\|S\|_p = \|T\|_p$, then $S = T$.*

Proof. For proof see [6]. □

Let $\mathcal{F}_1(H)$ be the set of all rank-one projections on the Hilbert space H.

Theorem 4.1. *Let $S \in \mathcal{L}(L^2_a(\mathbb{D}))$ be a positive operator. The following hold:*

(i) $\lim_{b\to\infty} (\|S + bE\| - b) = tr(SE)$ *for all* $E \in \mathcal{F}_1(L^2_a(\mathbb{D}))$, $b > 0$.

(ii) *If* $S \in S_p$, $1 < p < \infty$, *then* $\lim_{b\to\infty} (\|S + bE\|_p - b) = tr(SE)$ *holds for all* $E \in \mathcal{F}_1(L^2_a(\mathbb{D}))$, $b > 0$.

Proof. To prove (i), Suppose $f \in (\mathrm{Range}E) \cap L^2_a(\mathbb{D})$ with $\|f\| = 1$ and $\epsilon > 0$. Assume $T = (\langle Sf, f\rangle + \epsilon)E + bE^{\perp}$ where $E^{\perp} = I - E$. Then

$$T^{-1/2}ST^{-1/2} = \frac{1}{\langle Sf, f\rangle + \epsilon}ESE + \frac{1}{\sqrt{t}\sqrt{\langle Sf, f\rangle + \epsilon}}ESE^{\perp} +$$
$$\frac{1}{\sqrt{t}\sqrt{\langle Sf, f\rangle + \epsilon}}E^{\perp}SE + \frac{1}{t}E^{\perp}SE^{\perp} = \frac{1}{\langle Sf, f\rangle + \epsilon}ESE + V_b$$

where V_b is the sum of the last three terms. Notice that

$$ESEf = ES(\langle f, f\rangle f) = \langle f, f\rangle ESf = ESf = \langle Sf, f\rangle f.$$

Thus $\|ESEf\| = \langle Sf, f\rangle$. Hence

$$\|T^{-1/2}ST^{-1/2}\| \leq \frac{\langle Sf, f\rangle}{\langle Sf, f\rangle + \epsilon} + \|V_b\|.$$

Letting $b \longrightarrow \infty$, we obtain

$$\|T^{-1/2}ST^{-1/2}\| \leq \frac{\langle Sf, f\rangle}{\langle Sf, f\rangle + \epsilon} \leq 1$$

since $\|V_b\| \longrightarrow 0$ as $b \longrightarrow \infty$. Thus $T^{-1/2}ST^{-1/2} \leq I$ and therefore

$$S \leq (\langle Sf, f\rangle + \epsilon)E + bE^{\perp}.$$

Hence we obtain the inequality $\|(S+bE\| \leq \|\langle Sf,f\rangle+\epsilon+b)E+bE^{\perp}\|$ which holds for sufficiently large $b \geq 0$. Further, $\langle Sf,f\rangle+b \leq \|S+bE\|$ and

$$\|(\langle Sf,f\rangle+\epsilon+b)E+bE^{\perp}\| = \max\{\langle Sf,f\rangle+\epsilon+b, b\} = \langle Sf,f\rangle+\epsilon+b.$$

Thus for sufficiently large $b \geq 0$, we obtain

$$0 \leq \|S+bE\| - \langle Sf,f\rangle - b \leq \langle Sf,f\rangle+\epsilon+b-\langle Sf,f\rangle - b = \epsilon.$$

Hence we get,

$$\lim_{b\to\infty}(\|S+bE\|-b) = \langle Sf,f\rangle = \mathrm{tr}(SE).$$

To prove (ii), first notice that $\|bE\|_p = b$ and

$$\|S+bE\|_p - b = \frac{\|\frac{1}{b}S+E\|_p - \|E\|_p}{\frac{1}{b}}.$$

From [1], it follows that the Schatten-norm $\|\cdot\|_p$ is Fréchet differentiable at any point of $S_p(L^2_a(\mathbb{D}))$ and computing the derivative at the point E in the direction of S, we obtain

$$\begin{aligned}\lim_{b\to\infty}(\|S+bE\|_p - b) &= \lim_{b\to\infty}\frac{\|\frac{1}{b}S+E\|_p - \|E\|_p}{\frac{1}{b}}\\ &= \mathrm{tr}\left(\frac{|E|^{p-1}U^*S}{\|E\|_p^{p-1}}\right)\end{aligned}$$

where U is the partial isometry in the polar decomposition of E. Clearly, $U=E$, $|E|^{p-1} = E$ and $\|E\|_p = 1$ and hence we obtain that

$$\lim_{b\to\infty}(\|S+bE\|_p - b) = \mathrm{tr}(SE)$$

for $E \in \mathcal{F}_1(L^2_a(\mathbb{D}))$. □

Theorem 4.2. *Suppose $S,T \in \mathcal{L}(L^2_a(\mathbb{D}))$. The following hold:*

(i) *Suppose S is self-adjoint, $T \geq 0$ and $\pm S \leq T$. If further $S,T \in S_p$ for some p with $1 \leq p < \infty$ and $\|S\|_p = \|T\|_p$ then $\rho(|S|) = \rho(T)$.*

(ii) *If $S \geq 0$, $T \geq 0$, $S,T \in S_p$ for $1 < p \leq \infty$, then $\|S+bE\|_p \leq \|T+bE\|_p$ holds for all $b \geq 0$ and $E \in \mathcal{F}_1(L^2_a(\mathbb{D}))$ if and only if $S \leq T$. In this case, $\rho(S) \leq \rho(T)$.*

Proof. (i) Since $S = S^*$, the space $L^2_a(\mathbb{D})$ can be written as $L^2_a(\mathbb{D}) = X_+ \oplus X_-$ so that $S = \begin{pmatrix} S_+ & 0 \\ 0 & S_- \end{pmatrix}$, where S_+ and S_- are positive operators on X_+ and X_- respectively. Let $T = \begin{pmatrix} T_1 & T_2 \\ T_2^* & T_3 \end{pmatrix}$ relative to the decomposition $X = X_+ \oplus X_-$. Since $T \geq \pm S$, it follows that

$$\begin{pmatrix} T_1 - S_+ & T_2 \\ T_2^* & T_3 + S_- \end{pmatrix} \geq 0 \text{ and } \begin{pmatrix} T+S_+ & T_2 \\ T_2^* & T_3 - S_- \end{pmatrix} \geq 0. \tag{4.1}$$

Hence

$$T_1 \geq S_+ \text{ and } T_3 \geq S_-. \tag{4.2}$$

By [6], (4.2) and the min-max principle, we obtain

$$\|T\|_p \geq \left\|\begin{pmatrix} T_1 & 0 \\ 0 & T_3 \end{pmatrix}\right\|_p = \left(\|T_1\|_p^p + \|T_3\|_p^p\right)^{1/p} \geq \left(\|S_+\|_p^p + \|S_-\|_p^p\right)^{1/p} = \|S\|_p. \tag{4.3}$$

Now suppose that $\|S\|_p = \|T\|_p$. Then it follows from (4.2) and (4.3) that

$$\|T_1\|_p = \|S_+\|_p \text{ and } \|T_3\|_p = \|S_-\|_p. \tag{4.4}$$

From Lemma 4.1, it follows from (4.2) and (4.4) that $T_1 = S_+$ and $T_3 = S_-$. From (4.1), it follows that $T_2 = 0$ and so $\rho(T) = \rho(|S|)$.
(ii) Suppose $1 < p < \infty$ and assume $S \leq T$. It follows from the monotonicity of Schatten-p-norms [9] that $\|S + bE\|_p \leq \|T + bE\|_p$ for all $b \geq 0$ and for all $E \in \mathcal{F}_1(L^2_a(\mathbb{D}))$. Now assume that

$$\|S + bE\|_p \leq \|T + bE\|_p \tag{4.5}$$

holds for all $b \geq 0$ and for all $E \in \mathcal{F}_1(L^2_a(\mathbb{D}))$. From Theorem 4.1, it follows that $\lim_{b\to\infty} (\|S + bE\|_p - b) = \operatorname{tr}(SE)$. Thus we obtain from (4.5) that $\operatorname{tr}(SE) \leq \operatorname{tr}(TE)$ for all $E \in \mathcal{F}_1(L^2_a(\mathbb{D}))$. Thus it follows that

$$\langle Sf, f\rangle = \operatorname{tr}(S(f \otimes f)) \leq \operatorname{tr}(T(f \otimes f)) = \langle Tf, f\rangle$$

for all $f \in L^2_a(\mathbb{D})$ and $\rho(S) \leq \rho(T)$. For $p = \infty$, $S \leq T$ implies $\|S + bE\| \leq \|T + bE\|$ for all $b \geq 0$ and for all $E \in \mathcal{F}_1(L^2_a(\mathbb{D}))$. It follows from the monotonicity of the operator norm. Now suppose $\|S + bE\| \leq \|T + bE\|$ for all $t \geq 0$ and for all $E \in \mathcal{F}_1(L^2_a(\mathbb{D}))$. From Theorem 4.1, it follows that $\lim_{b\to\infty} (\|S + bE\| - b) = \operatorname{tr}(SE)$. Thus $\operatorname{tr}(SE) \leq \operatorname{tr}(TE)$. Hence

$$\begin{aligned}\langle Sf, f\rangle &= \lim_{b\to\infty} (\|S + b(f \otimes f)\| - b)\\ &\leq \lim_{b\to\infty} (\|T + b(f \otimes f)\| - b) = \langle Tf, f\rangle\end{aligned}$$

for all $f \in L^2_a(\mathbb{D})$. Therefore $S \leq T$ and $\rho(S) \leq \rho(T)$. The theorem follows. □

Definition 4.1. *An operator $A \in \mathcal{L}(L^2_a(\mathbb{D}))$ is a contraction if $\|A\| \leq 1$.*

Theorem 4.3. *Suppose $T \in \mathcal{L}(L^2_a(\mathbb{D}))$ is a contraction and $|T|^2 \leq |T^2|$. Then $\rho(K^{n+1}) \leq \rho(K^n)$ for all $n \in \mathbb{N}$ where $K = |T^2| - |T|^2$. Further $\{K^n\}$ converges strongly to a projection operator E and $\rho(K^n) \to \rho(E)$ and $\rho(TE) = 0$.*

Proof. Since $|T|^2 \leq |T^2|$, hence $K > 0$. Further, since $T \in \mathcal{L}(L^2_a(\mathbb{D}))$ it follows from [4] that $\||T|^{1/2}\|^2 = \|T\|$ and $\||T|f\| = \|Tf\|$ for all $f \in L^2_a(\mathbb{D})$. Let $S = K^{1/2}$ be the unique [4] non-negative square root of K. Now because T is a contraction, we obtain

$$\||T^2|^{1/2}\|^2 = \|T^2\| \leq 1.$$

Thus

$$\begin{aligned}\langle K^{n+1}f, f\rangle &= \|S^{n+1}f\|^2\\ &= \langle KS^n f, S^n f\rangle\\ &= \||T^2|^{1/2}S^n f\|^2 - \||T|S^n f\|^2\\ &\leq \|S^n f\|^2 - \|TS^n f\|^2 \leq \|S^n f\|^2 = \langle K^n f, f\rangle.\end{aligned}$$

Therefore $\langle K^{n+1}k_z, k_z\rangle \leq \langle K^n k_z, k_z\rangle$, for all $z \in \mathbb{D}$. That is $\rho(K^{n+1}) \leq \rho(K^n)$ for all $n \in \mathbb{N}$ and $\{K^n\}$ is a monotonically decreasing sequence of bounded positive operators. Now since $K \geq 0$, it follows from [2] that $\{K^n\}$ converges strongly to a projection E. Moreover,

$$\sum_{n=0}^{m} \|TS^n f\|^2 \leq \sum_{n=0}^{m} (\|S^n f\|^2 - \|S^{n+1}f\|^2) = \|f\|^2 - \|S^{m+1}f\|^2 \leq \|f\|^2$$

for all non-negative integers m and $f \in L^2_a(\mathbb{D})$. Therefore, $\|TS^n f\| \longrightarrow 0$ as $n \longrightarrow \infty$, and hence

$$TEf = T(\lim_{n\to\infty} K^n f) = \lim_{n\to\infty} TS^{2n} f = 0,$$

for every $f \in L^2_a(\mathbb{D})$. Thus $\rho(TE) = 0$. □

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PROXIMAL PLANAR SHAPE SIGNATURES. HOMOLOGY NERVES AND DESCRIPTIVE PROXIMITY

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Dedicated to J.H.C. Whitehead and Som Naimpally

ABSTRACT

This article introduces planar shape signatures derived from homology nerves,which are intersecting 1-cycles in a collection of homology groups endowed with a proximalrelator (set of nearness relations) that includes a descriptive proximity. A 1-cycle is a closed,connected path with a zero boundary in a simplicial complex covering a finite, boundedplanar shape. The signature of a shape shA (denoted by sig(shA)) is a feature vector thatdescribes shA. A signature sig(shA) is derived from the geometry, homology nerves, Bettinumber, and descriptive CW topology on the shape shA. Several main results are given,namely, (a) every finite, bounded planar shape has a signature derived from the homology group on the shape, (b) a homology group equipped with a proximal relator defines a descriptive Leader uniform topology and (c) a description of a homology nerve and unionof the descriptions of the 1-cycles in the nerve have same homotopy type.

1. INTRODUCTION

This paper introduces shape signatures restricted to the Euclidean plane. A finite, bounded planar shape A (denoted by shA) is a finite region of the Euclidean planebounded by a simple closed curve and with a nonempty interior [36].

After covering a shape with a simplicial complex,the signature of a shape is derived from the char acteristics of the simple closed connected pathsderived from connections between vertices in thecovering. A path in a simplicial complex is a se quence of connected simplexes. A closed path is aconnected path in which one can start at any vertex v in the path and traverse the path to reach

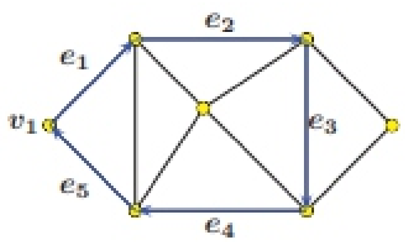

FIGURE 1. Path

v. A *simple closed path* contains no self intersections (loops). A pair of adjacent simplexes σ_1, σ_2 are *connected*, provided σ_1, σ_2 have a common part [10, §IV.1, p. 169].

A path is oriented, provided the path can be traversed in either forward (positive) or reverse (negative) direction. In other words, for any pair of adjacent edges in an oriented path, we can choose one of the edges and the direction to take in traversing the edges (*cf.*, M. Berger and G. Gostiaux [8, §0.1.3] and J.W. Ulrich [42, §2, p. 364] on oriented graphs).

Example 1. Sample Connected 1-simplexes in a Simple Closed Path.
Let e_1, e_2, e_3, e_4, e_5 be a sequence of oriented path containing 1-simplexes (edges) as shown in Fig. 1. The ordering of the 0-simplexes (vertices) is suggested by the directed edges. For example, $e_1 \to e_2 \to e_3 \to e_4 \to e_5 \to e_1$ defines a path. This path is closed, since $e_5 \to e_1$ at the end of a traversal of the edges, starting at v_1. This closed path is simple, since it has no loops.

A triangulated shape A (also denoted by sh$A$) is connected, provided there is an edge-wise simple closed path between each pair of vertices in sh$A$. Let K be a simplicial complex covering shape sh$A$. A 1-chain is a formal sum of edges leading from one vertex to another vertex on K. A 1-cycle is a 1-chain with an empty boundary. Also let σ_i denote the ith edge in a path in K, $C_1(K)$ be a set of cycles on edges on K and let $C_0(K)$ be a set of cycles on vertices on K. Let σ be a simplex spanned by the vertices $v_0, \dots, v_n$ in K. For $p \geq 1$, the homomorphic mapping $\partial_p : C_1(K) \longrightarrow C_0(K)$ is defined by

$$\partial_1 \sigma = \sum_{i=0}^{n} (-1)^i [v_0, \dots, v_n] = \sum_{i=0}^{n} \sigma_i.$$

The alternating signs on the terms indicate the simplexes are oriented, which means that for each positive term $+v_j$, there is a corresponding $-v_j, 0 \leq j \leq n$. The signs are inserted to take path orientation into account, so that all faces of a simplex are coherently oriented [19, §2.1].

The maps ∂_n are called *chain maps* (or *simplicial boundary maps*). Each chain map ∂_n is a *homomorphism*. The sum of the connected, oriented paths is called a *chain*. For a path with n edges in a triangulated planar shape, ∂_n defines a 1-chain. The vertices on a 1-simplex (edge) σ_i are the boundaries on σ_i. In other words, the boundary of n vertices $[v_0, \dots, v_n]$ is the $(n-1)$-chain formed by the sum of the faces [19, §2.1]. For a 1-chain $c = \sum \lambda_i \sigma_i, \lambda_i \in \mathbb{Z}$mod 2 (*i.e.*, for an integer coefficient λ_i in a 1-chain summand, λ_i mod 2 = 0 or 1), the *boundary* of the 1-chain is the sum of the boundaries of its 1-simplexes, namely,

$$\partial c = \lambda_1 \partial \sigma_1 + \cdots + \lambda_n \partial \sigma_n = \sum_{i=1}^{n} \lambda_i \partial \sigma_i.$$

Let K be a simplicial complex and let $C_2(K), C_1(K), C_0(K)$ be an additive Abelian group of 2-chains, 1-chains and 0-chains, respectively. Consider a sequence of homomorphisms (boundary maps) of Abelian groups, namely,

$$\cdots \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0.$$

Elements of img∂_2 are called boundaries. The quotient group $H_1 = Z_1/B_1 = \ker\partial_1/\text{img}\partial_2$ isolates those cycles in Z_1 with empty boundaries. Elements of H_1 are called 1-cycles, *i.e.*, those cycles in Z_1 that are not boundaries. From a quotient group perspective, elements of H_1 are cosets of img$\partial_2 = B_1$.

Let C_1 be a group of 1-chains of edges and let C_0 be a group of 0-chains of vertices. In general, p-chains under addition form an Abelian group (denoted by $(C_p, +)$ or $C_p = C_p(K)$, when addition is understood). Each member of C_0 is a 0-chain (a linear combination of vertices) on the boundary of a 1-chain in C_1. The kernel $\partial_1 : C_1(K) \longrightarrow C_0(K)$ is a group denoted by Z_1. Elements of ker∂_1 are called cycles. The image of ∂_2 is the group $B_1 = B_1(K)$, which is a subgroup of Z_1.

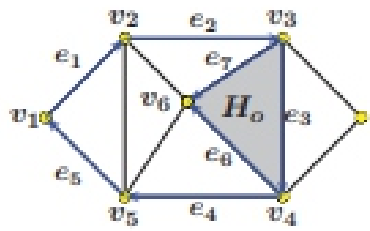

FIGURE 2. **1-cycle**

Example 2. Sample Cycles.
For example, let edges $e_1, e_2, e_3, e_4, e_5, e_6, e_7$ and vertices $v_1, v_2, v_3, v_4, v_5, v_6$ on a triangulated shape (not shown) be represented in Fig. 2. Then, we have

$\boldsymbol{B_1}$: *collection of boundaries written as 1-chains,* e.g.,
- $\partial(e_3, e_6, e_7) = \partial H_o = v_3 + v_4 - v_6$ *is the boundary of the hole H_o in Fig. 2.*

$\boldsymbol{Z_1}$: *collection of cycles written as 1-chains. For simplicity, we consider only three cycles in Z_1 based on the labelled edges in Fig. 2, namely,*
- $\partial(e_1, e_2, e_3, e_4, e_5) = v_1 + v_2 + v_3 + v_4 + v_5 - v_5 - v_4 - v_3 - v_2 - v_1 = 0.$
- $\partial(e_1, e_2, e_7, e_6, e_4, e_5) = v_1 + v_2 + v_3 + v_6 + v_4 + v_5 - v_5 - v_4 - v_6 - v_4 - v_3 - v_2 - v_1 = 0.$
- $\partial(e_3, e_6, e_7) = \partial H_o = v_3 + v_4 - v_6$ *(appears in B_1).*

Remark 1.1. *With the quotient group H_1, we factor out of Z_1 the chains that are the hole boundaries in B_1. From the features of the 1-cycles in homology groups H_1, we define a signature of a shape based on the description of 1-cycles, which is easily compared with the signatures of other shapes.*

Let $(\mathcal{H}_1, \delta_\Phi)$ be a collection of 1-cycles on shape complexes equipped with a descriptive proximity δ_Φ [12, §4], [32, §1.8], based on the descriptive intersection $\underset{\Phi}{\cap}$ of nonempty sets A and B [28, §3]. With respect to 1-cycle sets of connected, oriented edges e_1, e_2 in H_1, for example, we consider $e_1 \underset{\Phi}{\cap} e_2$. For each given 1-cycle A (denoted by cyc$A$), find all 1-cycles cyc$B$ in $\mathcal{H}_1$ that have nonempty descriptive intersection with cyc$A$, *i.e.*, cyc$A \underset{\Phi}{\cap}$ cyc$B \neq \emptyset$. This results in a Leader uniform topology on H_1 [23] and a main result in this paper.

Let $A \overset{\wedge\!\wedge}{\delta} B$ be a strong proximity between nonempty sets A and B, *i.e.*, A and B have nonempty intersection.

Theorem 1.1. *Let $\left(\mathcal{H}_1, \left\{\overset{\wedge\!\wedge}{\delta}, \delta_\Phi\right\}\right)$ be a collection of 1-dimensional homology groups H_1 equipped with a proximal relator $\left\{\overset{\wedge\!\wedge}{\delta}, \delta_\Phi\right\}$ and which is a collection of 1-cycles on a simplicial complex covering a finite, bounded planar shape and let*

$$\Phi(\mathcal{H}_1) = \{\Phi(cycA) : \text{1-cycle } cycA \in \mathcal{H}_1\} \ \ \textbf{(Set of descriptions of cyc}\boldsymbol{A \in \mathcal{H}_1}\textbf{)}$$

be a set of descriptions $\Phi(cycA)$ of 1-cycles cyc$A$ in $\mathcal{H}_1$. A Leader uniform topology is derivable from $\Phi(\mathcal{H}_1)$.

This section briefly presents the basic approach to defining finite, bounded planar shape barcodes based on two useful proximities (strong spatial proximity ∧∧δ and descriptive proximity δΦ). A shape barcode is a feature vector that describes a specific shape interms of 1-cycle geometry, rank of H1, characteristics of a homology nerve on H1, closurefiniteness and 1-cycle arc characteristics based on a descriptive weak topology on H1. Byproximity of a pair of sets, we mean spatial closeness of the sets. For a complete introduction to spatial proximity, see A. Di Concilio [13] and the earlier overview of proximityby S.A. Naimpally and B.D. Warrack [26]. A proximal hit-and-miss topology is a naturaloutcome of the traditional forms of proximity (see, e.g., G. Beer [5, §2.2, p. 45]). By descriptive proximity of a pair of sets, we mean the closeness of the descriptions of the sets.For a complete study of descriptive proximity, see A. Di Concilio, C. Guadagni, J.F. Petersand S. Ramanna [12]. In Section 2.5, a descriptive CW topology (Closure finite Weaktopology) is defined for a collection H1 of homology groups H1 equipped the descriptiveproximity δΦ.

2.1. Basic Approach.

The basic approach in homology in classifying a finite, bounded planar shape shA covered with a simplicial complex K is to analyze a collection H1 ofhomology groups H1 on shA, which is a set of 1-cycles. A 1-cycle A in H1 (denotedby cycA) is a simple, closed, connected path containing 1-simplexes (edges) that arenot boundaries of holes in shA. The story starts by identifying 1-dimensional homologygroups Z1 (i.e., groups whose members are cycles that are closed, connected paths on 1-simplexes) and 1-dimensional groups B1 containing cycles that are boundaries of holes.From Z1 and B1, we then derive a homology group H1 = Z1/B1 (a quotient group whichfactors out the cycle boundaries in Z1) containing 1-cycles.Notice that every planar shape has a distinguished 1-cycle, namely, the contour of ashape. The features (distinguishable characteristics) of 1-cycles in H1 provide a barcodefor a particular shape shA, which is a feature vector in an n-dimensional Euclidean spaceRn. A shape shA barcode describes shA and is an instance of the signature of the shape(denoted by sig(shA)). In the study of a shape shA that persists and yet changes over time,the rank of H1 is an important shape characteristic to include in the signature sig(shA). Insimple terms, the rank of H1 is the number of 1-cycles in H1 [6, §2.2, p. 96] on complexK on a shape shA. The rank of H1 (denoted by rH1) is also called the Betti numberof H1. Viewing the rank of H1 in another way, the Betti number of H1 is the numberZ summands, when H1 is written as the direct sum of its cyclic subgroups [19, §2.1, p.1390]. For example, the rank of Z1 for Example 2 is 2.

2.2. Framework for Two Recent Proximities.

This section briefly presents a framework for two recent types of proximities, namely, strong proximity and the more recent descriptive proximity in the study of computational proximity [32].Let A be a nonempty set of vertices, p ∈ A in a bounded region X of the Euclideanplane. An open ball Br(p) with radius r is defined by

$$Br(p) = \{q \in X : \|p - q\| < r\} \text{ (Open ball with center p, radius r).}$$

The closure of A (denoted by clA) is defined by

$$clA = \{q \in X : Br(q) \subset A \text{ for some } r\} \text{ (Closure of set A).}$$

The boundary of A (denoted by bdyA) is defined by

$$bdyA = \{q \in X : B(q) \subset A \cap X \setminus A\} \text{ (Boundary of set A)}$$

Of great interest in the study of shapes is the interior of a shape, found by subtracting the boundary of a shape from its closure. In general, the *interior* of a nonempty set $A \subset X$ (denoted by int$A$) defined by

$$\text{int}A = \text{cl}A - \text{bdy}A \text{ (Interior of set } A).$$

Proximities are nearness relations. In other words, a *proximity* between nonempty sets is a mathematical expression that specifies the closeness of the sets. A *proximity space* results from endowing a nonempty set with one or more proximities. Typically, a proximity space is endowed with a common proximity such as the proximities from Čech [41], Efremovič [15], Lodato [24], and Wallman [44], or the more recent descriptive proximity [29].

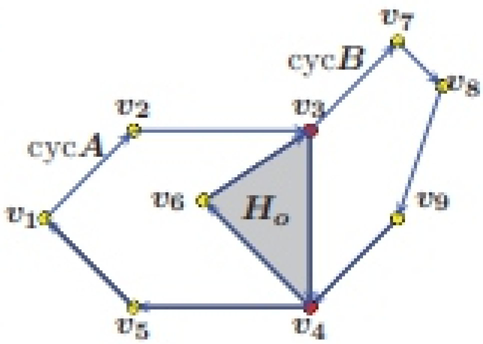

FIGURE 3. cyc$A$ $\overset{\doublewedge}{\delta}$ cyc$B$

2.3. **Strong Proximity.** Nonempty sets A, B in a space X equipped with the strong proximity $\overset{\doublewedge}{\delta}$ are *strongly near* [*strongly contacted*] (denoted $A \overset{\doublewedge}{\delta} B$), provided the sets have at least one point in common.L The strong contact relation $\overset{\doublewedge}{\delta}$ was introduced in [31] and axiomatized in [38], [18, §6 Appendix] (see, also, [32, §1.5], [31, 37]) and elaborated in [32].

Let $A, B, C \subset X$ and $x \in X$. The relation $\overset{\doublewedge}{\delta}$ on the family of subsets 2^X is a *strong proximity*, provided it satisfies the following axioms.

(snN0): $\emptyset \overset{\doublewedge}{\not\delta} A, \forall A \subset X$, and $X \overset{\doublewedge}{\delta} A, \forall A \subset X$.

(snN1): $A \overset{\doublewedge}{\delta} B \Leftrightarrow B \overset{\doublewedge}{\delta} A$.

(snN2): $A \overset{\doublewedge}{\delta} B$ implies $A \cap B \neq \emptyset$.

(snN3): If $\{B_i\}_{i \in I}$ is an arbitrary family of subsets of X and $A \overset{\doublewedge}{\delta} B_{i^*}$ for some $i^* \in I$ such that int$(B_{i^*}) \neq \emptyset$, then $A \overset{\doublewedge}{\delta} (\bigcup_{i \in I} B_i)$

(snN4): int$A \cap$ int$B \neq \emptyset \Rightarrow A \overset{\doublewedge}{\delta} B$.

When we write $A \overset{\doublewedge}{\delta} B$, we read A is *strongly near* B (A *strongly contacts* B). The notation $A \overset{\doublewedge}{\not\delta} B$ reads A is not strongly near B (A does not *strongly contact* B). For each *strong proximity* (*strong contact*), we assume the following relations:

(snN5): $x \in \text{int}(A) \Rightarrow x \overset{\doublewedge}{\delta} A$

(snN6): $\{x\} \overset{\doublewedge}{\delta} \{y\} \Leftrightarrow x = y$

For strong proximity of the nonempty intersection of interiors, we have that $A \overset{\doublewedge}{\delta} B \Leftrightarrow$ int$A \cap$ int$B \neq \emptyset$ or either A or B is equal to X, provided A and B are not singletons; if $A = \{x\}$, then $x \in \text{int}(B)$, and if B too is a singleton, then $x = y$. It turns out that if $A \subset X$ is an open set, then each point that belongs to A is strongly near A. The bottom line is that strongly near sets always share points, which is another way of saying that sets with strong contact have nonempty intersection.

Example 3. *Assume that a finite, bounded shape shA is covered by a simplicial complex containing 1-cycles $cycA, cycB$. Let 1-cycle $cycA$ be represented by a sequence of vertices*

$$v_1 \to v_2 \to v_3 \to v_6 \to v_4 \to v_5 \to v_1 \text{ (cycA)},$$

and let 1-cycle $cycB$ be represented by a sequence of vertices

$$v_3 \to v_7 \to v_8 \to v_9 \to v_4 \to v_6 \to v_3 \text{ (cycB)},$$

as shown in Fig. 3. Notice, for example, that the interior of 1-cycle $cycA$ includes the arc $\widehat{v_3v_6}$, which is also in the interior of 1-cycle $cycB$. In this case, $int(cycA), int(cycB)$ have $\widehat{v_3v_6}$ in common. Hence, from axiom (snN4), $cycA \ \overset{\text{m}}{\delta}\ cycB$.

Definition 2.1. *Let K be a simplicial complex covering a shape shA and let $\mathcal{H}_1$ be the collection of 1-cycles in the homology groups on K. A* homology nerve *on $\mathcal{H}_1(K)$ (denoted by $Nrv\mathcal{H}_1$) is defined by*

$$Nrv\mathcal{H}_1 = \left\{ cycA \in \mathcal{H}_1 : \bigcap cycA \neq \emptyset \right\} \ \textbf{(Homology Nerve)}.$$

The assumption made here is that every finite planar shape is bounded by a simple closed curve and has a nonempty interior.

Conjecture 2.1. *Every finite, bounded, planar shape with a decomposition and with at least one hole contains a homology nerve that intersects with the boundary of a hole.*

Conjecture 2.2. *Every finite, bounded, planar shape with a decomposition and with at least one hole contains a homology nerve that does not intersect with the boundary of any hole.*

Remark 2.1. Short History of Topological Nerves.
In topology, a nerve structure first appeared in 1926 in a paper on simplicial approximation by P. Alexandroff [3] and in 1932 in a monograph by P. Alexandroff [2, §, p. 39], elaborated by C. Kuratowski in 1933 [22]. Let the system of sets $F_1, \dots, F_s$ and system of vertices $v_1, \dots, v_s$ of a complex K be related in such a way that the sets $F_{i_1}, \dots, F_{r_p}$ have nonempty intersection if and only if the vertices $v_{i_0}, \dots, v_{i_r}$ belong to K. Then the complex K is called the nerve of the system of sets in K. A fundamental theorem concerning simplicial nerve complexes is given by B. Grünbaum in 1970 [17], namely,

Theorem 2.1. *Each simplicial complex has the same homotopy type as its nerve.*

Earlier, K. Borsuk obtained the following result in 1948.

Theorem 2.2. [9, Cor. 2, p. 233] *Finite dimensional spaces admitting similar regular decompositions have necessarily the same homotopy type.*

As a result of Theorem 2.2, K. Borsuk observed that (i) for every finite dimensional space with a regular decomposition, there exists a polytope with the same homotopy type and (ii) the notion of an Alexandroff nerve makes it possible to construct such a polytope [9, p. 233], leading to

Corollary 2.1. [9, Cor. 3, p. 234] *If the simplicial complex K is a geometrical realization of the nerve of a regular decomposition of a finite dimensional space A, then the space A and the polytope $|K|$ have the same homotopy type.*

A more tractable view of a nerve, more amenable for computational topology, is given by H. Edelsbrunner and J.L. Harer [14, §III.2, p. 59]. Let F be a finite collection of sets. A nerve *consists of all nonempty subcollections of F (denoted by $NrvF$) whose sets have nonempty intersection*, i.e.,

$$NrvF = \left\{ X \subseteq F : \bigcap X \neq \emptyset \right\} \text{ (Edelsbrunner-Harer Nerve).}$$

A nerve is an example of an abstract simplicial complex, regardless of the sets in F. Strongly proximal Edelsbrunner-Harer nerves were introduced in 2016 by J.F. Peters and E. İnan [39]. *Nerve spoke complexes (useful for nerves on Voronoï tessellations) are introduced in J.F. Peters* [35]. *An overview of recent work on nerve complexes is given by H. Dao, J. Doolittle, K. Duna, B. Goeckner, B. Holmes and J. Lyle* [11].

Example 4. *Assume that a shape shA is covered by a simplicial complex with homology groups H_1 containing 1-cycles $cycA, cycB$ from Example 3. Hence,*

$$Nrv\mathcal{H}_1 = \{cycA, cycB\} \ \textbf{(Sample homology nerve)}.$$

Lemma 2.1. *Let homology groups H_1 contain 1-cycles $cycA, cycB$ on complex K covering shape shA. Then $cycA \overset{\wedge\wedge}{\delta} cycB \Rightarrow cycA \cap cycB \neq \emptyset$, if and only if $cycA, cycB \in Nrv\mathcal{H}_1$ for homology nerve complex $Nrv\mathcal{H}_1 \in 2^{\mathcal{H}_1}$.*

Proof. $\mathrm{cyc}A \overset{\wedge\wedge}{\delta} \mathrm{cyc}B \Rightarrow \mathrm{cyc}A \cap \mathrm{cyc}B \neq \emptyset$ (from (snN2)) $\Leftrightarrow \mathrm{cyc}A, \mathrm{cyc}B \in \mathrm{Nrv}\mathcal{H}_1$ (from Def. 2.1) for at least one nerve complex $\mathrm{Nrv}\mathcal{H}_1 \in 2^{\mathcal{H}_1}$. □

Lemma 2.2. *Let $Nrv_1\mathcal{H}_1, Nrv_2\mathcal{H}_1$ be homology nerves for homology groups H_1 on complex K covering shape shA. Then $Nrv_1\mathcal{H}_1 \overset{\wedge\wedge}{\delta} Nrv_2\mathcal{H}_1$ implies $Nrv_1\mathcal{H}_1 \cap Nrv_2\mathcal{H}_1 \neq \emptyset$ for some $cycA \in Nrv_1\mathcal{H}_1$ and $cycB \in Nrv_2\mathcal{H}_1$.*

Proof. $\mathrm{Nrv}_1\mathcal{H}_1 \overset{\wedge\wedge}{\delta} \mathrm{Nrv}_2\mathcal{H}_1 \Rightarrow \mathrm{Nrv}_1\mathcal{H}_1 \cap \mathrm{Nrv}_2\mathcal{H}_1 \neq \emptyset$ (from (snN2)). Consequently, $\mathrm{cyc}A \overset{\wedge\wedge}{\delta} \mathrm{cyc}B \rightarrow \mathrm{cyc}A \cap \mathrm{cyc}B \neq \emptyset$ (from Lemma 2.1) for at least one $\mathrm{cyc}A \in \mathrm{Nrv}\mathcal{H}_1$ and for at least one $\mathrm{cyc}B \in \mathrm{Nrv}\mathcal{H}_2$, since a homology nerve is a set of 1-cycles (from Def. 2.1). □

Theorem 2.3. *Let $Nrv_1\mathcal{H}_1, Nrv_2\mathcal{H}_1$ be homology nerves for homology groups H_1 on a simplicial complex covering shape shA. $Nrv_1\mathcal{H}_1 \overset{\wedge\wedge}{\delta} Nrv_2\mathcal{H}_1$ if and only if $cycA \overset{\wedge\wedge}{\delta} cycB$ for some $cycA \in Nrv_1\mathcal{H}_1$ and $cycB \in Nrv_2\mathcal{H}_1$.*

Proof. Immediate from Lemma 2.2. □

Corollary 2.2. *Let $Nrv_1\mathcal{H}_1, Nrv_2\mathcal{H}_1, Nrv_3\mathcal{H}_1$ be homology nerves for homology groups H_1 on a simplicial complex covering shape shA. If $(Nrv_1\mathcal{H}_1 \cup Nrv_2\mathcal{H}_1) \overset{\wedge\wedge}{\delta} Nrv_3\mathcal{H}_1$, then*

$$Nrv_1\mathcal{H}_1 \overset{\wedge\wedge}{\delta} Nrv_3\mathcal{H}_1$$

or

$$Nrv_2\mathcal{H}_1 \overset{\wedge\wedge}{\delta} Nrv_3\mathcal{H}_1$$

for the three homology nerves $Nrv_1\mathcal{H}_1, Nrv_2\mathcal{H}_1, Nrv_3\mathcal{H}_1$ on the homology groups H_1.

Proof. From Lemma 2.2, $(\mathrm{Nrv}_1\mathcal{H}_1 \cup \mathrm{Nrv}_2\mathcal{H}_1) \overset{\wedge\wedge}{\delta} \mathrm{Nrv}_3\mathcal{H}_1 \Rightarrow (\mathrm{Nrv}_1\mathcal{H}_1 \cup \mathrm{Nrv}_2\mathcal{H}_1) \cap \mathrm{Nrv}_2\mathcal{H}_1 \neq \emptyset$. And, from Theorem 2.3, $(\mathrm{Nrv}_1\mathcal{H}_1 \cup \mathrm{Nrv}_2\mathcal{H}_1) \overset{\wedge\wedge}{\delta} \mathrm{Nrv}_3\mathcal{H}_1$ if and only if $\mathrm{cyc}A \overset{\wedge\wedge}{\delta} \mathrm{cyc}B$ for some $\mathrm{cyc}A \in (\mathrm{Nrv}_1\mathcal{H}_1 \cup \mathrm{Nrv}_2\mathcal{H}_1)$ and $\mathrm{cyc}B \in \mathrm{Nrv}_3\mathcal{H}_1$. Hence, $\mathrm{Nrv}_1\mathcal{H}_1 \overset{\wedge\wedge}{\delta} \mathrm{Nrv}_3\mathcal{H}_1$ or $\mathrm{Nrv}_2\mathcal{H}_1 \overset{\wedge\wedge}{\delta} \mathrm{Nrv}_3\mathcal{H}_1$. □

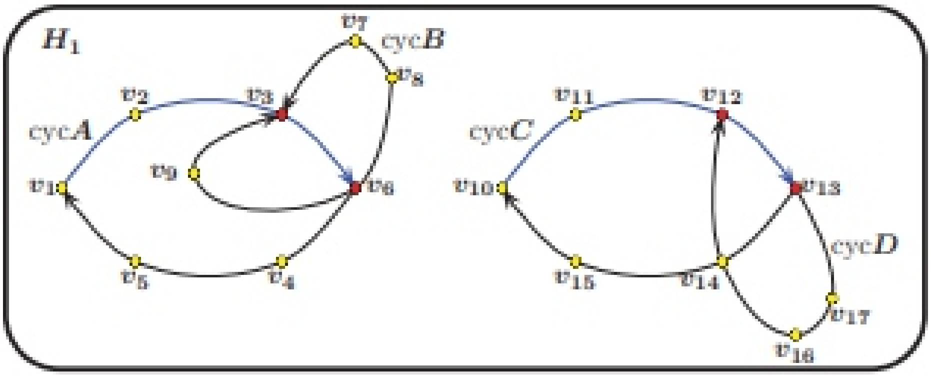

FIGURE 4. cyc$A \overset{\Phi}{\delta}$ cyc$B$ and cyc$A\ \delta_\Phi$ cyc$C$

2.4. **Descriptive Proximity.** In the run-up to a close look at extracting features from shape complexes, we first consider descriptive proximities introduced in [29], fully covered in [12] and briefly introduced, here. Descriptive proximities resulted from the introduction of the descriptive intersection of pairs of nonempty sets [29], [25, §4.3, p. 84].

(Φ): $\Phi(A) = \{\Phi(x) \in \mathbb{R}^n : x \in A\}$, set of feature vectors.

($\underset{\Phi}{\cap}$): $A \underset{\Phi}{\cap} B = \{x \in A \cup B : \Phi(x) \in \Phi(A) \& \in \Phi(x) \in \Phi(B)\}$.

Let $\Phi(x)$ be a feature vector for an arc x in a simplicial complex on a planar shape. For example, let $\Phi(x)$ be a feature vector representing single arc feature such as a Fourier descriptor in measuring the difference between arcs in a complex [27] or uniform iso-curvature of arc along a curved edge [7, §2.2]. For simplicity, we limit the description of an arc to the uniform iso-curvature of the arc between vertices in the curved edges of a 1-cycle such as those shown in Fig. 4. $A\ \delta_\Phi\ B$ reads A is descriptively near B, provided $\Phi(x) = \Phi(y)$ for at least one pair of points, $x \in A, y \in B$. The proximity δ in the Čech, Efremovič, and Wallman proximities is replaced by δ_Φ, which satisfies the following Descriptive Lodato Axioms from [30, §4.15.2].

(dP0): $\emptyset\ \not\delta_\Phi\ A, \forall A \subset X$.
(dP1): $A\ \delta_\Phi\ B \Leftrightarrow B\ \delta_\Phi\ A$.
(dP2): $A \underset{\Phi}{\cap} B \neq \emptyset \Rightarrow\ A\ \delta_\Phi\ B$.
(dP3): $A\ \delta_\Phi\ (B \cup C) \Leftrightarrow A\ \delta_\Phi\ B$ or $A\ \delta_\Phi\ C$.
(dP4): $A\ \delta_\Phi\ B$ and $\{b\}\ \delta_\Phi\ C$ for each $b \in B\ \Rightarrow A\ \delta_\Phi\ C$ (**Descriptive Lodato**).

Proposition 2.1. [36, §2.2] *Let (X, δ_Φ) be a descriptive proximity space, $A, B \subset X$. Then $A\ \delta_\Phi\ B \Rightarrow A \underset{\Phi}{\cap} B \neq \emptyset$.*

Proof. See [36, §2.2] for the proof. □

Next, consider a proximal form of a Száz relator [40]. A *proximal relator* $\mathcal{R}$ is a set of relations on a nonempty set X [33]. The pair $(X, \mathcal{R})$ is a proximal relator space. The connection between $\overset{\Phi}{\delta}$ and δ is summarized in Prop. 2.3.

Lemma 2.3. [36, §2.2] *Let $\left(X, \left\{\delta_\Phi, \overset{\Phi}{\delta}\right\}\right)$ be a proximal relator space, $A, B \subset X$. Then $A\ \overset{\Phi}{\delta}\ B \Rightarrow A\ \delta_\Phi\ B$.*

Proof. See [36, §2.2] for the proof. □

Example 5. Descriptively Near 1-Cycles in H_1. *Let $cycA, cycB, cycC, cycD$ in Fig. 4 be 1-cycles in a collection of homology groups $\mathcal{H}_1$ on a simplicial complex covering a planar shape. Further, for example, let*

$$\Phi(cycA) = \left\{\Phi(\widehat{vv'}) \in cycA : \Phi(\widehat{vv'}) = \text{ uniform iso-curvature of } \widehat{vv'}\right\}.$$

Let $\mathcal{H}_1$ be equipped with the relator $\left\{\overset{\text{sn}}{\delta}, \delta_\Phi\right\}$. Then observe

1^o *In Fig. 4,*

$$cycA \text{ has } \overbrace{[v_1, v_2, v_3, v_4, v_5, v_6]}^{cycA \text{ vertices}} : v_1 \to v_2 \to v_3 \to v_6 \to v_4 \to v_5 \to v_1.$$
$$cycB \text{ has } \overbrace{[v_3, v_6, v_7, v_8]}^{cycB \text{ vertices}} : v_3 \to v_6 \to v_8 \to v_7 \to v_3.$$
$$\text{edge } \widehat{v_3v_6} \in int(cycA) \text{ and } \widehat{v_3v_6} \in int(cycB), \text{ i.e., } int(cycA) \cap int(cycB) \neq \emptyset.$$

Consequently, from Axiom (snN4), $cycA \overset{\text{sn}}{\delta} cycB$ and from Lemma 2.3, $cycA\ \delta_\Phi\ cycB$. Hence, from Proposition 2.1, $cycA \underset{\Phi}{\cap} cycB \neq \emptyset$.

2^o *In Fig. 4,*

$$cycC \text{ has } \overbrace{[v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}]}^{cycC \text{ vertices}} :$$
$$v_{10} \to v_{11} \to v_{12} \to v_{13} \to v_{14} \to v_{15} \to v_{10}.$$
$$cycD \text{ has } \overbrace{[v_{12}, v_{13}, v_{17}, v_{16}, v_{14}]}^{cycD \text{ vertices}} : v_{12} \to v_{13} \to v_{17} \to v_{16} \to v_{14} \to v_{12}.$$
$$cycA\ \delta_\Phi\ cycC \text{ , since } \overbrace{\Phi(\widehat{v_3v_6}) = \Phi(\widehat{v_{12}, v_{13}})}^{\text{arcs have matching uniform iso-curvature}}.$$

Consequently, $cycA \underset{\Phi}{\cap} cycC \neq \emptyset$. From Axiom (dP2), $cycA\ \delta_\Phi\ cycC$. Hence, from Proposition 2.1, the converse also holds, i.e.,

$$cycA\ \delta_\Phi\ cycC \Rightarrow cycA \underset{\Phi}{\cap} cycC \neq \emptyset.$$

In other words, the 1-cycles $cycA, cycC$ in homology groups $\mathcal{H}_1$ represented in Fig. 4 have descriptive proximity, since $cycA, cycC$ have curved edges with the same uniform iso-curvature.

Let $2^{2^{\mathcal{H}_1}}$ denote a collection of sub-collections of 1-cycles $\mathcal{H}_1$.

Theorem 2.4. *Let $\left(\mathcal{H}_1, \left\{\delta_\Phi, \overset{\text{sn}}{\delta}\right\}\right)$ be a collection of homology groups endowed with a proximal relator and let 1-cycles $cycA, cycB \in \mathcal{H}_1$, homology nerves $Nrv_1\mathcal{H}_1, Nrv_2\mathcal{H}_1 \in 2^{2^{\mathcal{H}_1}}$. Then*

1^o *$Nrv_1\mathcal{H}_1 \overset{\text{sn}}{\delta} Nrv_2\mathcal{H}_1$ implies $Nrv_1\mathcal{H}_1\ \delta_\Phi\ Nrv_2\mathcal{H}_1$.*
2^o *A 1-cycle $cycA \in Nrv_1\mathcal{H}_1 \cap Nrv_2\mathcal{H}_1$ implies $cycA \in Nrv_1\mathcal{H}_1 \underset{\Phi}{\cap} Nrv_2\mathcal{H}_1$.*

3^o *$cycA \overset{\text{sn}}{\delta} cycB \to cycA\ \delta_\Phi\ cycB$.*

Proof.
1^o: Immediate from Lemma 2.3.
2^o: Let $cycA \in \mathcal{H}_1$. $\text{cyc}A \in \text{Nrv}_1\mathcal{H}_1 \cap \text{Nrv}_2\mathcal{H}_1$, provided $\text{Nrv}_1\mathcal{H}_1 \overset{\text{\tiny m}}{\delta} \text{Nrv}_2\mathcal{H}_1$. Then $\text{cyc}A \in \text{Nrv}_1\mathcal{H}_1 \underset{\Phi}{\cap} \text{Nrv}_2\mathcal{H}_1$. Hence, from Prop. 2.1, $\text{Nrv}_1\mathcal{H}_1\ \delta_\Phi\ \text{Nrv}_2\mathcal{H}_1$.
3^o: Immediate from Lemma 2.3. □

Corollary 2.3. *Let $\left(\mathcal{H}_1, \left\{\delta_\Phi, \overset{\text{\tiny m}}{\delta}\right\}\right)$ be a collection of homology groups endowed with proximal relator, homology nerves $Nrv_1\mathcal{H}_1, Nrv_2\mathcal{H}_1 \in 2^{2^{\mathcal{H}_1}}$ with $Nrv_2\mathcal{H}_1$ on shape shB. Then*

1^o *$Nrv_1\mathcal{H}_1 \overset{\text{\tiny m}}{\delta} Nrv_2\mathcal{H}_1$ implies $Nrv_1\mathcal{H}_1\ \delta_\Phi\ shB$.*
2^o *$Nrv_1\mathcal{H}_1 \cap Nrv_2\mathcal{H}_1 \neq \emptyset$ implies $Nrv_1\mathcal{H}_1 \underset{\Phi}{\cap} Nrv_2\mathcal{H}_1$.*

2.5. **Descriptive Homology Nerves and Shape Signature.** This section introduces descriptive homology nerves and the components of a shape signature.

Definition 2.2. *Let K be a simplicial complex covering a shape shA and let $\mathcal{H}_1$ be the collection of 1-cycles in homology groups H_1 on K.* A descriptive homology nerve on $\mathcal{H}_1(K)$ *(denoted by $Nrv_\Phi\mathcal{H}_1$) is defined by*

$$Nrv_\Phi\mathcal{H}_1 = \left\{cycA \in \mathcal{H}_1 : \bigcap_\Phi cycA \neq \emptyset\right\} \text{ (\textbf{Descriptive Homology Nerve})}.$$

The nucleus of a descriptive homology nerve is any member $cycA \in Nrv_\Phi\mathcal{H}_1$ that serves as a representative of the nerve inasmuch as $cycA$ defines a cluster X that contains all 1-cycles $cycB$ such that $cycA\ \delta_\Phi\ cycB$.

Theorem 2.5. *Let K be a simplicial complex covering a finite, bounded planar shape, $\mathcal{H}_1$ a collection of homology groups on K, and $\Phi(\mathcal{H}_1)$ a set of descriptions of the 1-cycles in $\mathcal{H}_1$. Every member of $\Phi(\mathcal{H}_1)$ is the nucleus of a descriptive homology nerve $Nrv_\Phi\mathcal{H}_1$.*

Proof.
By definition, $\Phi(\mathcal{H}_1) = \{\Phi(\text{cyc}A) : \text{cyc}A \in \mathcal{H}_1\}$. Let $\Phi(\text{cyc}A) \in \Phi(\mathcal{H}_1)$. Since $\text{cyc}A\ \delta_\Phi\ \text{cyc}A$ then, from Def. 2.2, $\text{cyc}A$ is the nucleus of a descriptive nerve $\text{Nrv}_\Phi\mathcal{H}_1$ containing one cycle, namely, $\text{cyc}A$. Let

$$X = \{\text{cyc}B \in \mathcal{H}_1 : \text{cyc}A\ \delta_\Phi\ \text{cyc}B \neq \emptyset\}.$$

Hence, by Def. 2.2, X is a descriptive homology nerve and $\text{cyc}A$ is the nucleus of the nerve X, i.e., $\text{cyc}A\ \delta_\Phi\ \text{cyc}B$ for every member $\text{cyc}B \in X$. □

Example 6. *Let the collection of homology groups $\mathcal{H}_1$ be represented the 1-cycles $cycA$, $cycB$, $cycC$, $cycD$ in Fig. 4. Let uniform iso-curvature be used to describe a 1-cycle in $\mathcal{H}_1$. Notice that curved edge $\widehat{v_3v_6} \in cycB$ has the same uniform iso-curvature as $\widehat{v_{12}v_{13}} \in cycC$ and $\widehat{v_{12}v_{13}} \in cycD$. Hence,*

$$Nrv_\Phi\mathcal{H}_1 = \{cycB, cycC, cycD \in \mathcal{H}_1\} \text{ (\textbf{Descriptive Homology Nerve})},$$

since

$$\bigcap_{\substack{cycX \in \\ \{cycB, cycC, cycD\}}} cycX \neq \emptyset.$$

From Theorem 2.5, $cycB$ is the nucleus of $Nrv_\Phi\mathcal{H}_1$.

Conjecture 2.3. Every finite, bounded, planar shape with a decomposition and with at least one hole contains a descriptive homology nerve that intersects with the boundary of a hole.

Conjecture 2.4. Every finite, bounded, planar shape with a decomposition and with at leastone hole contains a descriptive homology nerve that does not intersect with the boundary ofany hole.Consider next a basis for a shape signature.

Definition 2.3. Shape Signature.Let H1 be a collection of homology groups on a simplicial complex covering a shape shA, a finite bounded planar region with nonempty interior and let Nrv1H1,Nrv2H1 ∈ 2 H1. Assumethat H1 is equipped with a proximal relator {∧ ∧δ , δΦ}. A signature of shape shA (denoted bysig(shA)) is a feature vector that includes at least one of the following components.
1o Geometry: One or more features of the curvature of each 1-cycle cycA ∈ H1 are included in sig(shA) that describes shape shA.

2 o Homology: rank of the homology group H1 (denoted by rH1), i.e., number of 1- cycle generators of H1 is defined in terms of the rank of the cycles group Z1 (denotedby rZ1) and the rank of the boundaries group B1 (denoted by rB1) . Recall thatrH1 = r(Z1/B1) = rZ1 − rB1 (Rank of a homology group) [43, p. 63].The rank rH1 (a Betti number) can change over time and provides a useful inindicator of planar shape persistence. Hence, its inclusion in a shape shA signaturesig(shA) (barcode) is important in considering the persistent topology of data suchas that found in R. Ghrist [16].

3o Homology Nerve: Since every cycA ∈ H1 is the nucleus of a descriptive homology nerve NrvΦH1 (from Theorem 2.5), select a component of Φ(cycA) (call it x)with a description that matches the description of the same component in the othermembers of NrvΦH1. Include Φ(x) in the signature of shA, i.e.,sig(shA) = (. . . , Φ(x). . .) (Φ(x) in feature vector that describes shA).

4o Closure Finiteness: Let vvc′ be an arc in a 1-cycle cycA ∈ H1 and cl(vvc′) intersects only a finite number of other arcs in H1. cl(vvc′) is the closure of an arcin cycA ∩ΦcycB for a finite number of 1-cycles. For cycA,cycB ∈ H1, chooseΦ(cl(vvc′)) ∈ sig(shA) or Φ(cycA) ∈ sig(shA) for a selected number of 1-cycles inH1.

5o descriptive CW: (i.e., descriptive Weak Topology) Assume that Closure Finiteness holds for the collection of homology groups H1 equipped with the descriptiveproximity δΦ. Let vvc′ be an arc in H1 ∈ H1 and let 1-cycle cycA ∈ H1. Then cycA closed in H1, provided cycA ∩ vvc′⊭ is also closed in H1 . Then cycA ∧ ∧δ vvc′.Hence, from Lemma 2.3, cycA δΦ vvc′. For example, 1-cycles cycA,cycB in Fig.

4 overlap, since arc vd3v6 is common to both 1-cycles. Such arcs provide an incisive feature for a shape signature. Then, for a shape shA, include the description of sucharcs in the shape signature sig(shA).

Remark 2.2. The original idea of a CW topology (Closure finite Weak topology) was toshift from structures in simplicial complexes K that are the focus in P. Alexandroff [2] andin P. Alexandroff, H. Hopf [4] to homological structures called cells and cell complexes (e.g.,0-cells (vertices) and 1-cells (open arcs) attached to a shape skeleton via maps to obtain a cell complex) in a homology on K [45, p. 214]. A cell complex is a finite collection of cells [19]. With a descriptive CW, we shift from a description of structures (e.g., simplicialnerves [34, p. 2] and nerve spokes [34, §2.2, p. 4] [1, Def. 9, p. 8]) in simplicial complexesto a description of structures such as homology nerves, collections of 1-cycles and overlap ping arcs in a collection of homology groups H1 in cell complexes on finite bounded planarshapes. Basically, with a descriptive CW on H1, we include those features of arcs, 1-cycles andhomology nerves in H1 that provide a complete signature sig(shA) for a shape shA. The motivation for doing this is an interest in measuring the persistence of the feature values of arcs,1-cycles and homology nerves in homology groups over time. This descriptive CW is based onthe Closure finiteness and Weak topology axioms for a traditional CW complex given by K.Jänich [21, §VII.3, p. 95] founded on its original introduction by J.H.C. Whitehead [45].

3. MAIN RESULT

Theorem 3.1. Every finite, bounded planar shape shA covered by a simplicial complex has a signature derived from the homology group on the complex.Proof. From Def. 2.3, it is enough to include the rank of H1 in sig(shA) for a shape shAto have a signature.

Lemma 3.1. Let H1 be a collection of homology groups equipped with the proximal relator
RΦ ={∧∧δ , δΦ}on a simplicial complex covering a finite, bounded shape. Every collection of1-dimensional homology groups H1 ∈ H1 endowed with the proximal relator RΦ defines adescriptive uniform Leader topology on H1.
Proof. The basic approach in this proof is to use the steps for constructing a uniform topology introduced by S. Leader [23] in constructing a descriptive uniform topology.∩Φ: For each Nrv1H1 ∈ H1, select all Nrv2H1 ∈ H1 such that Nrv1H1∧∧δ Nrv2H1, i.e., thepair of homology nerves Nrv1H1 ∈ H1 overlap (have strong proximity). From Lemma 2.3,Nrv1H1 ∩ΦNrv2H1 ⊭ . Hence, Nrv1H1 ∩ΦNrv2H1 ∈ Φ(H1).∪Φ: By definition,

Nrv1H1 ∪ΦNrv2H1 = {cycA ∈ H1 : cycA ∈ Nrv1H1
∩ΦNrv2H1or Φ(cycA) ∈ Φ(Nrv1H1) or Φ(cycA) ∈ Φ(Nrv2H1)}

Hence, Nrv1H1 ∪ Φ Nrv2H1 ∈ Φ(H1).

Remark 3.1. Lemma 3.1 is a stronger result than we need to derive a descriptive CW, which is a convenient setting for the study of finite, bounded planar shapes signatures. Theorem 1.1is a direct result of Lemma 3.1.

Theorem 3.2. [14, §III.2, p. 59] Let F be a finite collection of closed, convex sets in Euclidean space. Then the nerve of F and the union of the sets in F have the same homotopytype.

Lemma 3.2. Let H1 be a collection of homology groups on a simplicial complex covering afinite, bounded shape. Then a homology nerve NrvH1 ∈ 2H1 and ∪cycA∈H1cycA have samehomotopy type

Proof. H1 is a collection of 1-cycles, which are closed, convex sets in Euclidean space. Then from Theorem 3.2, NrvH1 and ∪cycA∈H1cycA have same homotopy type.

Theorem 3.3. Let (H1,{∧∧δ , δΦ}) be a collection of homology groups H1 equipped with aproximal relator on a simplicial complex covering a finite, bounded shape. Then Φ(NrvH1)) ∈2Rn(a description of a homology nerve) and ∪Φ(cycA)∈Φ(NrvH1)Φ(cycA) (union of the descrip tions) have same homotopy type.

Proof. Each member of Φ(H1) is feature vector in Rn and each point in Rn is a closed,convex singleton set. Then from Lemma 3.2, Φ(NrvH1) and ∪Φ(cycA)∈Φ(NrvH1)Φ(cycA)have same homotopy type.

Remark 3.2. Open Problems.Let shA be a finite, bounded planar shape covered with a simplicial complex K and let H1(K)be a homology group on K.

An open problem in shape theory is selecting each 1-cycle that is the contour of a subshapecontaining a hole in shA.

A second open problem in shape theory is the construction of a collection of homologynerves that overlap a subshape of interest in a shape shA.Let H1(K) be a collection of homology groups on a simplicial complex K.

A third openproblem in shape theory is detecting space curves (also called twisted curves by D. Hilbertand S. Cohn-Vossen [20, §27]) overlapping with 1-cycles in H1(K).

A fourth open problem in shape theory is to use homology nerves as a basis for measuringthe persistence over time of object shapes in digital images.

A fifth open problem in shape theory is to measure the persistence of a finite, boundedshape over time using a shape signature that includes the uniform iso-curvature of the 1-cycles and the Betti number of a homology group on the shape

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