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# MATHEMATICS AND STATISTICS

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## Contents

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# Aspects of Algebraic Structure of Rough Sets

**S. Sangeetha, Shakeela Sathish**
Department of Mathematics, SRM Institute of Science and Technology, Ramapuram Campus, India

## ABSTRACT

Rough sets are extensions of classical sets characterized by vagueness and imprecision. The main idea of rough set theory is to use incomplete information to approximate the concept of imprecision or uncertainty, or to treat ambiguous phenomena and problems based on observation and measurement. In Pawlak rough set model, equivalence relations are a key concept, and equivalence classes are the foundations for lower and upper approximations.

Developing an algebraic structure for rough sets will allow us to study set theoretic properties in detail. Several researchers studied rough sets from an algebraic perspective and a number of structures have been developed in recent years, including rough semigroups, rough groups, rough rings, rough modules, and rough vector spaces. The purpose of this study is to demonstrate the usefulness of rough set theory in group theory. There have been several papers investigating the roughness in algebraic structures by substituting an algebraic structure for the universe set. In this paper, rough groups are defined using upper and lower approximations of rough sets from a finite universe instead of considering the whole universe. Here we have considered a finite universe $\Lambda$ along with a relation $\chi$ which classifies the universe into equivalence classes. We have identified all rough sets with respect to this relation. The upper and lower approximated sets have been taken separately and these form a rough group equivalence relation ($\chi_{rog}$) and it partitions the group $(2^{\Lambda},\triangle)$ into equivalence classes. In this paper, the rough group approximation space $(2^{\Lambda},\chi rog)$ has been defined along with upper and lower approximations and properties of subsets of $2^{\Lambda}$ with respect to rough group equivalence relations have been illustrated.

**Keywords** Rough Group, Rough Group Approximation Space, Rough Group Equivalence Relation

## 1 Introduction

There are numerous mathematical concepts that are delivered through the use of set theory, which is used as a core method in the entire field of mathematics. Pawlak introduced the concept of rough sets [1]. There has been an increase in interest in this newly emerging theory in recent years. Since its introduction, the rough set theory has been continued to develop as a tool for classifying.

The rough set has been evaluated algebraically by a variety of experts to date. The topics of interest range from pure theory, such as topological and algebraic foundations, to applications as discussed in [2], [3], & [4]. The concept of rough sets has been approached algebraically by Bonikowski [5], Iwinski [6] & Pomykala [7]. A rough subring is defined by Davvaz [8] when rough set theory and ring theory are considered. The rough group has been evaluated by N. Kuroki and Wang [9] in order to approximate the upper and lower bounds of any subset of a group in terms of its normal subgroup. In addition, topological rough groups were defined and their properties were examined in [10]. A generalized rough set can be viewed in two different ways according to Radwan et al [11]. Based on a family of dominance relationships, Salama et al [12] gave properties of different types of rough approximations. A topological approach was given by Al-Shami [13] to generate new rough set models. Also using E-neighborhoods, Al-Shami[14] provided new rough approximations. A topological approach to rough approximations based on closure operators was developed by El-Bably et al [15]. Through ideals, Guler et al [16] provided rough approximations based on different topologies. The concept of generalized rough approximation spaces based on maximal neighbourhoods and ideals is discussed by Hosny et al [17]. Several types of rough sets based on coverings were provided by Nawar et al [18]. Based on j-neighborhood space and j-adhesion neighborhood space, Atef et al [19] compared six types of rough approximations. Using J-Nearly Concepts via Ideals Hosny [20] gave a topological approach for rough sets. Pradeep Shende et al [21] presented a novel concept of uncertainty optimization through multi-granular rough sets. A rough set with uncertainty optimization based on incomplete information systems was introduced by Arvind et al [22].

In particular, Biswas [23] introduced rough groups and rough subgroups. Miao et al [24] modified the approach by proving the group axioms to the upper approximation of a set. Wang [25] examined the relationship between the normal series of a group and its rough approximations in order to determine the properties of rough groups.

In this paper, modified approach to rough groups is provided followed by a rough group approximation space based on rough group equivalences.

The rough set theory has been briefly reviewed in section 2. In section 3, we explored rough groups from a different perspective. Wehave introduced the rough group approximation space in section 4 and given the upper and lower approximations of any set based on rough group equivalence relations. The significance of this work is presented in results and discussion section 5.

## 2 Basics of Rough Set Theory

**Definition 2.1 [1]**

Approximation space is composed of a finite set univ1 (= ϕ) and ”ι1”, an equivalence relation on univ1 and it is represented by (univ1,”ι1”).

**Definition 2.2 [1]**

Afamily of subsets E = {E1,E2,E3......En} of univ1 are said to be a classification of univ1 if

- E1∪E2∪.....∪En = univ1
- Ei ∩Ej =ϕ,fori= j

**Definition 2.3 [1]**

Let (univ1,”ι1”) be an approximation space and for any k ∈ univ1 the set [k]”ι1” is called the equivalence class induced by ”ι1”.

**Definition 2.4 [1]**

Consider K = (univ1,”ι1”), an approximation space and A be any subset of univ1 then

- univ1A = {ai | [ai]”ι1” ∩ A= ϕ}
- univ1A = {ai | [ai]”ι1” ⊆ A}
- BNA=univ1A−univ1A

are called approximations of upper, lower & boundary regions of A in relation to χ respectively and if the boundary of the set A is not empty, it is said to be rough, otherwise it is said to be crisp.

If A,B ⊆ univ1, then the following results are due to [1]

- univ1A ⊆ A ⊆univ1A
- univ1univ1 = univ1univ1 = univ1
- univ1A∪B = univ1A ∪univ1B
- univ1A∪B ⊇ univ1A ∪univ1B
- univ1A∪B = univ1A ∪univ1B
- univ1A∩B ⊆ univ1A ∩uiv1Bn
- If A ⊆Bthenuniv1A ⊆ univ1B &univ1A ⊆ univ1B

## 3 Rough Groups

**Definition 3.1 : Group[26]**

Groups are non-empty sets with binary operations that satisfy closure, associativity, identity, and inverse properties under .

**Definition 3.2 : Power Set[26]**

Collection of all possible subsets of G forms a Power set represented by 2G which forms an abelian group along with operation △

**Definition 3.3 [1]**

(U,R1), an approximation space. R1, an equivalence relation partitions U into classes of equivalence. Let W(= ϕ) ⊆ U. R1W ={w | [w]R1 ∩W=ϕ},whichisupper approximation of W R1W ={w| [w]R1 ⊆ W} which is lower approximation of W if RW −RW=ϕ then W = (RW,RW) is a rough set otherwise crisp

**Definition 3.4**

(U,R), an approximation space which consists of a finite set U of n elements. ($2^U$,△) forms an abelian group and R(U), a collection of all rough sets in U is said to be a rough group if R(U)△R(U) with respect to the binary operation △ forms subgroup of ($2^U$,△)

**Theorem 3.1**

Let R(U) represents all possible rough sets in a space (U,R). R(U)∪R(U) with respect to the binary operation △ forms subgroup of ($2^U$,△) and hence R(U) is said to be rough group.

**Proof 1**

Let R(U) be the set of all rough sets of (U,R).

R(U) = RU = {

W|W ∈R(U)}

R(U) = RU = {W

|W ∈R(U)}

Let us denote R(rog) = RU ∈RU

To prove R(rog), subgroup of 2U.

R(rog) is non empty since ϕ is always a subset of any set and it will be in RU

Let W1,W2 ∈ R(rog)

Claim : W1△W2 = (W1 ∪W2)−(W1∩W2)∈R(rog)

∵W1,W2 ∈R(rog)

W1,W2 ∈ RU ∪RU

W1,W2∈RUorW1,W2∈RU

**Case1**

IfW1,W2∈RU

W1=B1&W2=B2whereB1,B2∈R(U)

W1△W2=(B1∪B2)−(B1∩B2)

B1∪B2={[u]R|[u]R∩B1∪B2=ϕ}

={[u]R|[u]R∩B1∪[u]R∩B2=ϕ}=⇒B1∪B2∈R(U)

B1∩B2={[u]R|[u]R∩B1∩B2=ϕ}

={[u]R|[u]R∩B1∩[u]R∩B2=ϕ} =⇒B1∩B2∈R(U)

AssumingW2⊆W1

(B1∪B2)−(B1∩B2)={[u]R∩B1=ϕ or [u]R∩B2=

ϕ}−{[u]R∩B1=ϕ and [x]R∩B2=ϕ}

={[u]R∩B1=ϕ or [u]R∩B2=ϕ}=B1∪B2∈R(U)

=⇒W1△W2∈RU

**Case2**

IfW1,W2∈RU

W1=B1&W2=B2whereB1,B2∈R(U)

W1△W2=(B1∪B2)−(B1∩B2)

B1∪B2⊇(B1∪B2)∈R(U)

B1∩B2=(B1∩B2)

AssumeB2⊆B1

(B1∩B2)∈R(U)

B1∪B2={[u]R|[u]R⊆B1∪B2 =⇒[u]R∩B1∪B2=[u]R}

B1∩B2={[u]R|[u]R⊆B1∩B2 =⇒[u]R∩B1∩B1=[u]R}

(B1∪B2)−(B1∩B2)=[u]R∩B1∪B2∈R(U)

W1△W2∈RU

Hence if W1,W2∈RU∪RU,thenW1△W2∈RU∪RU.

Also all elements poss self inverse .Hence RU ∪ RU is a sub group of 2U and hence R(U) is said to be Rough group.

R(U)={{1},{2},{1,3},{2,3}}

R(U)={{1,2},{1,2,3}}

R(U)={{},{3}}

R(rog)=R(U)∪R(U)={{},{3},{1,2},{1,2,3}} Hence

{} {3} {1,2} {1,2,3}

{} {} {3} {1,2} {1,2,3}

{3} {3} {} {1,2,3} {1,2}

{1,2} {1,2} {1,2,3} {} {3}

{1,2,3} {1,2,3} {1,2} {3} {}

R(rog) is a sub group of 2U and hence R(U) is a Rough Group.

**Theorem3.2** If”Эa”(rog),”Эb”(rog)are rough groups, then ”Эa”(rog)∩”Эb”(rog)is also rough group. (uni1 is finite universe)

**Proof**

Let x,y∈”Эa”(rog)∩”Эb”(rog)

Since ϕ⊆anyset.soϕ∈”Эa”(uni1)∩”Эb”(uni1)

=⇒ϕ∈”Эa”(rog)∩”Эb”(rog)

=⇒”Эa”(rog)∩”Эb”(rog)=ϕ

Since”Эa”(rog)&”Эb”(rog)are sub groups of 2uni1

=⇒x△y∈”Эa”(rog)∩”Эb”(rog)

Hence the result.

**Proposition3.1**

Let”Эa”rog &”Эb”rog be two rough groups then ”Эa”rog ∩”Эb”rog ⊆”Эa”rog ∩”Эb”rog (where”Эa”&”Эb” are two equivalence relations,uni1is the universe)

***Proof***

$$\begin{aligned}
\overline{''\partial_a''_{ro_g} \cap ''\partial_b''_{ro_g}} &= \overline{[\overline{''\partial_a''(uni_1)} \cup \underline{''\partial_a''(uni_1)}] \cap [\overline{''\partial_b''(uni_1)} \cup \underline{''\partial_b''(uni_1)}]} \\
&= \overline{[\overline{''\partial_a''(uni_1)} \cap \overline{''\partial_b''(uni_1)}] \cup [\underline{''\partial_a''(uni_1)} \cap \underline{''\partial_b''(uni_1)}]} \\
&= \overline{[\overline{''\partial_a''(\kappa_1)} \cap \overline{''\partial_b''(uni_1)}]} \cup \overline{[\underline{''\partial_a''(uni_1)} \cap \underline{''\partial_b''(uni_1)}]} \\
&\subseteq [\overline{\overline{''\partial_a''(uni_1)}} \cap \overline{\overline{''\partial_b''(uni_1)}}] \cup [\overline{\underline{''\partial_a''(uni_1)}} \cap \overline{\underline{''\partial_b''(uni_1)}}] \\
&= [\overline{''\partial_a''(uni_1)} \cap \overline{''\partial_b''(uni_1)}] \cup [\underline{''\partial_a''(uni_1)} \cap \underline{''\partial_b''(uni_1)}] \\
&= [\overline{''\partial_a''(uni_1)} \cup \underline{''\partial_a''(uni_1)}] \cap [\overline{''\partial_b''(uni_1)} \cup \underline{''\partial_b''(uni_1)}] \\
&\subseteq ''\partial_a''ro_g \cap ''\partial_b''ro_g, \text{ hence upper approximation of intesection of rough groups is contained in its intersection}
\end{aligned}$$

B

**Proposition 3.2** *Let* $''\partial_a''_{ro_g}$ & $''\partial_b''_{ro_g}$ *be two rough groups then*
$\overline{''\partial_a''_{ro_g} \cup ''\partial_b''_{ro_g}} = ''\partial_a''_{ro_g} \cup ''\partial_b''_{ro_g}$

***Proof***

$$\begin{aligned}
\overline{''\partial_a''_{r_g} \cup ''\partial_b''_{r_g}} &= \overline{[\overline{''\partial_a''(uni_1)} \cup \underline{''\partial_a''(uni_1)}] \cup [\overline{''\partial_b''(uni_1)} \cup \underline{''\partial_b''(uni_1)}]} \\
&= \overline{[\overline{''\partial_a''(uni_1)} \cup \overline{''\partial_b''(uni_1)}] \cup [\underline{''\partial_a''(uni_1)} \cup \underline{''\partial_b''(uni_1)}]} \\
&= \overline{[\overline{''\partial_a''(uni_1)} \cup \overline{''\partial_b''(uni_1)}]} \cup \overline{[\underline{''\partial_a''(uni_1)} \cup \underline{''\partial_b''(uni_1)}]} \\
&= [\overline{\overline{''\partial_a''(uni_1)}} \cup \overline{\overline{''\partial_b''(uni_1)}}] \cup [\overline{\underline{''\partial_a''(uni_1)}} \cup \overline{\underline{''\partial_b''(uni_1)}}] \\
&= [\overline{''\partial_a''(uni_1)} \cup \overline{''\partial_b''(uni_1)}] \cup [\underline{''\partial_a''(uni_1)} \cup \underline{''\partial_b''(uni_1)}] \\
&= [\overline{''\partial_a''(uni_1)} \cup \underline{''\partial_a''(uni_1)}] \cup [\overline{''\partial_b''(uni_1)} \cup \underline{''\partial_b''(uni_1)}] \\
&= ''\partial_a''_{ro_g} \cup ''\partial_b''_{ro_g}, \textit{ hence upper approximation of union of rough groups is equal to its union}
\end{aligned}$$

B

**Proposition 3.3** *Let* $''\partial_a''_{ro_g}$ & $''\partial_b''_{ro_g}$ *are two rough groups then*
$\underline{''\partial_a''_{ro_g} \cap ''\partial_b''_{r_g}} = ''\partial_a''_{ro_g} \cap ''\partial_b''_{ro_g}$

***Proof:***

$$\begin{aligned}
\underline{''\partial_a''_{ro_g} \cap ''\partial_b''_{ro_g}} &= \underline{[\overline{''\partial_a''(uni_1)} \cup \underline{''\partial_a''(uni_1)}] \cap [\overline{''\partial_b''(uni_1)} \cup \underline{''\partial_b''(uni_1)}]} \\
&= \underline{[\overline{''\partial_a''(uni_1)} \cap \overline{''\partial_b''(uni_1)}] \cup [\underline{''\partial_a''(uni_1)} \cap \underline{''\partial_b''(uni_1)}]} \\
&= \underline{[\overline{''\partial_a''(uni_1)} \cap \overline{''\partial_b''(uni_1)}]} \cup \underline{[\underline{''\partial_a''(uni_1)} \cap \underline{''\partial_b''(uni_1)}]} \\
&= [\underline{\overline{''\partial_a''(uni_1)}} \cap \underline{\overline{''\partial_b''(uni_1)}}] \cup [\underline{\underline{''\partial_a''(uni_1)}} \cap \underline{\underline{''\partial_b''(uni_1)}}] \\
&= [\overline{''\partial_a''(uni_1)} \cap \overline{''\partial_b''(uni_1)}] \cup [\underline{''\partial_a''(uni_1)} \cap \underline{''\partial_b''(uni_1)}] \\
&= [\overline{''\partial_a''(uni_1)} \cup \underline{''\partial_a''(uni_1)}] \cap [\overline{''\partial_b''(uni_1)} \cup \underline{''\partial_b''(uni_1)}] \\
&= ''\partial_a''_{ro_g} \cap ''\partial_b''_{ro_g}, \textit{ hence lower approximation of intesection of rough groups is equal to its intersection}
\end{aligned}$$

# 4 Rough Group Approximation Space

**Definition 4.1**

***Rough Groups Relation***

*Let* $(uni_1, ''\gamma_1'')$ *be approximation space &* $G = (2^{uni_1}, \triangle)$ *be a group & an equivalence relation* $''\gamma_1''(ro_g)$ *partitions* $G$ *into equivalence classes* $\{''\gamma_1''(uni_1), \overline{''\gamma_1''(uni_1)}, \underline{''\gamma_1''(uni_1)}\}$. *Then the space* $(2^{uni_1}, \triangle)$ *is called rough group approximation*

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*space. For $A \subseteq 2^{uni_1}$, the upper & lower approximations are given by*

$$"\gamma_1"(ro_g)^A = \{A_1 \in 2^{uni_1} | [A_1]_{"\gamma_1"(ro_g)} \cap A \neq \phi\}$$

$$"\gamma_1"(ro_g)_A = \{A_1 \in 2^{uni_1} | [A_1]_{"\gamma_1"(ro_g)} \subseteq A\}$$

**Proposition 4.1**
*Let $A, B \subseteq 2^{uni_1}$ be nonempty and $"\gamma_1"(ro_g)$ be a rough group equivalence relation then*

1. $"\gamma_1"(ro_g)_A \subseteq A \subseteq "\gamma_1"(ro_g)^A$
2. $"\gamma_1"(ro_g)^{A\cup B} = "\gamma_1"(ro_g)^A \cup "\gamma_1"(ro_g)^B$
3. $"\gamma_1"(ro_g)_{A\cap B} = "\gamma_1"(ro_g)_A \cap "\gamma_1"(ro_g)_B$
4. $A \subseteq B \implies "\gamma_1"(ro_g)_A \subseteq "\gamma_1"(ro_g)_B$
5. $A \subseteq B \implies "\gamma_1"(ro_g)^A \subseteq "\gamma_1"(ro_g)^B$
6. $"\gamma_1"(ro_g)_{A\cup B} \supseteq "\gamma_1"(ro_g)_A \cup "\gamma_1"(ro_g)_B$
7. $"\gamma_1"(ro_g)^{A\cap B} \subseteq "\gamma_1"(ro_g)^A \cap "\gamma_1"(ro_g)^B$
8. *$"\gamma_1"(ro_g)$ and $"\gamma_1"(ro_g)$ are equivalence relations then* $"\gamma_1"(ro_g) \subseteq "\gamma_1"(ro_g) \implies "\gamma_1"(ro_g)^A \subseteq "\gamma_1"(ro_g)^A$

**Proof:**

1. Let $X_1 \in "\gamma_1"(ro_g)_A$
   $\implies X_1 \in [Y_1]_{"\gamma_1"(ro_g)}$
   also $[Y_1]_{"\gamma_1"(ro_g)} \subseteq A \implies X_1 \in A$

   $X_1 \triangle \phi \subseteq X\underline{"\gamma_1"(uni_1)}$ ($\because \phi \in \underline{"\gamma_1"(uni_1)}$)

   $X_1 \in X_1\underline{"\gamma_1"(uni_1)} \cap A$

   $\implies X_1 \in "\gamma_1"(ro_g)^A$

2. To prove
   $"\gamma_1"(ro_g)^{A_1\cup B_1} = "\gamma_1"(ro_g)^{A_1} \cup "\gamma_1"(ro_g)^{B_1}$

$$\begin{aligned}X \in "\gamma_1"(ro_g)^{A_1\cup B_1} &\Leftrightarrow X \in [Y]_{"\gamma_1"(ro_g)} \cap (A_1 \cup B_2)\\ &\Leftrightarrow X \in ["\gamma_1"(uni_1) \cup \overline{("\gamma_1"(uni_1)} \cup \underline{("\gamma_1"(uni_1))}] \cap (A_1 \cup B_1)\\ &\Leftrightarrow X \in ["\gamma_1"(uni_1) \cap A_1] \cup X \in [\overline{"\gamma_1"(uni_1)} \cup \underline{"\gamma_1"(uni_1)} \cap B]\\ &\Leftrightarrow X \in "\gamma_1"(ro_g)^{A_1} \text{ or } X \in "\gamma_1"(ro_g)^{B_1}\\ &\Leftrightarrow X \in "\gamma_1"(ro_g)^{A_1} \cup X \in "\gamma_1"(ro_g)^{B_1}\\ &\Leftrightarrow X \in "\gamma_1"(ro_g)^{A_1} \cup "\gamma_1"(ro_g)^{B_1}\end{aligned}$$

3. To Prove $"\gamma_1"(ro_g)_{A\cap B} = "\gamma_1"(ro_g)_A \cap "\gamma_1"(ro_g)_B$

$$\begin{aligned}X \in "\gamma_1"(ro_g)_{(A\cap B)} &\Leftrightarrow X \in [Y]_{"\gamma_1"(ro_g)} \subseteq (A \cap B)\\ &\Leftrightarrow X \in ["\gamma_1"(uni_1) \cup \overline{("\gamma_1"(uni_1)} \cup \underline{("\gamma_1"(uni_1))}] \subseteq (A \cap B)\\ &\Leftrightarrow X \in ["\gamma_1"(uni_1) \cup \overline{("\gamma_1"(uni_1)} \cup \underline{("\gamma_1"(uni_1))}]\\ &\subseteq A \cap ["\gamma_1"(uni_1) \cup \overline{("\gamma_1"(uni_1)} \cup \underline{("\gamma_1"(uni_1))}] \subseteq B)\\ &\Leftrightarrow X \in "\gamma_1"(ro_g)_A \cap "\gamma_1"(ro_g)_B\end{aligned}$$

4. To prove $A \subseteq B \implies "\gamma_1"(ro_g)_A \subseteq "\gamma_1"(ro_g)_B$
   Since $A \cap B = A$
   $"\gamma_1"(ro_g)_A = "\gamma_1"(ro_g)_{(A\cap B)} = "\gamma_1"(ro_g)_A \cap "\gamma_1"(ro_g)_B$
   $\implies "\gamma_1"(ro_g)_A \subseteq "\gamma_1"(ro_g)_B$

5. To prove
   $A \subseteq B \implies "\gamma_1"(ro_g)^A \subseteq "\gamma_1"(ro_g)^B$
   Since $A \cup B = B$
   $"\gamma_1"(ro_g)^B = "\gamma_1"(ro_g)^{(A\cup B)} = "\gamma_1"(ro_g)^A \cup "\gamma_1"(ro_g)^B$
   $\implies "\gamma_1"(ro_g)^A \subseteq "\gamma_1"(ro_g)^B$

6. To prove
   $"\gamma_1"(ro_g)_{(A_1\cup B_1)} \supseteq "\gamma_1"(ro_g)_{A_1} \cup "\gamma_1"(ro_g)_{B_1}$
   $A_1 \cup B_1 \supseteq A_1$ and $B_1$
   $"\gamma_1"(ro_g)_{A_1} \subseteq "\gamma_1"(ro_g)_{(A_1\cup B_1)}$
   $"\gamma_1"(ro_g)_{B_1} \subseteq "\gamma_1"(ro_g)_{(A_1\cup B_1)}$
   Hence $"\gamma_1"(ro_g)_{(A_1\cup B_1)} \supseteq "\gamma_1"(ro_g)_{A_1} \cup "\gamma_1"(ro_g)_{B_1}$

7. To prove
   $"\gamma_1"(ro_g)^{A_1\cap B_1} \subseteq "\gamma_1"(ro_g)^{A_1} \cap "\gamma_1"(ro_g)^{B_1}$
   since $A_1 \supseteq A_1 \cap B_1$ & $A_1 \cap B_1 \subseteq B_1$
   then $"\gamma_1"(ro_g)^{A_1\cap B_1} \subseteq "\gamma_1"(ro_g)^{A_1} \cap "\gamma_1"(ro_g)^{B_1}$

8. Let $\eth1_{ro_g}$ & $\eth2_{ro_g}$ be two rough group relations on $(2^{\kappa_1}, \triangle)$
   To prove $\eth1_{ro_g} \subseteq \eth2_{ro_g} \implies \eth1^A_{ro_g} \subseteq \eth2^A_{ro_g}$
   $\eth1_{ro_g} \subseteq \eth2_{ro_g} \implies \eth1(\kappa_1) \cup \overline{\eth1(\kappa_1)} \cup \underline{\eth1(\kappa_1)} \subseteq \eth2(\kappa_1) \cup \overline{\eth2(\kappa_1)} \cup \underline{\eth2(\kappa_1)}$
   $\implies \eth1^A_{ro_g} \subseteq \eth2^A_{ro_g}$

**Proposition 4.2** *Let $"\gamma_1(ro_g)"$ and $"\gamma_2"(ro_g)$ be rough group equivalence relations on $G = (2^{uni_1}, \triangle)$ then* $("\gamma_1"(ro_g) \cap "\gamma_2"(ro_g))^A = ("\gamma_1"(ro_g))^A \cap ("\gamma_2"(ro_g))^A$

***Proof***

$$\begin{aligned}X_1 \in ("\gamma_1"(ro_g) \cap "\gamma_2"(ro_g))^A &\Leftrightarrow X_1 \in [y_1]_{("\gamma_1"(ro_g) \cap "\gamma_2"(ro_g))} \cap A\\ &\Leftrightarrow X_1 \in ["\gamma_1"(uni_1) \cap "\gamma_1"(uni_1)] \cup [\overline{"\gamma_1"(uni_1)} \cap \overline{"\gamma_2"(uni_1)}] \cup [\underline{"\gamma_2""(uni_1)} \cap \underline{"\gamma_2"(uni_1)}] \cap A\\ &\Leftrightarrow X_1 \in ["\gamma_1"(uni_1) \cup \overline{"\gamma_1"(uni_1)} \cup \underline{"\gamma_1"(uni_1)}] \cap ["\gamma_2"(uni_1) \cup \overline{"\gamma_2"(uni_1)} \cup \underline{"\gamma_2"(uni_1)}] \cap A\\ &\Leftrightarrow X_1 \in [["\gamma_1"(uni_1) \cup \overline{"\gamma_1"(uni_1)} \cup \underline{"\gamma_1"(uni_1)}] \cap A] \cap [["\gamma_2"(uni_1) \cup \overline{"\gamma_2"(uni_1)} \cup \underline{"\gamma_2"(uni_1)}] \cap A]\\ &\Leftrightarrow X_1 \in ("\gamma_1"(ro_g))^A \cap ("\gamma_2"(ro_g))^A\end{aligned}$$

*Hence the upper approximation of any set with respect to intersection of rough group equivalence relation is equal to intersection of upper approximation of the set.*

B

**Proposition 4.3**

*Let* $"\gamma_1(ro_g)"$ *and* $"\gamma_2"(ro_g)$ *be rough group equivalence relations on* $G = (2^{\kappa_1}, \triangle)$ *then* $("\gamma_1"(ro_g) \cap "\gamma_2"(ro_g))_A = ("\gamma_1"(ro_g))_A \cap ("\gamma_2"(ro_g))_A$

***Proof:***

$$\begin{aligned}
X_1 \in ("\gamma_1"(ro_g) \cap "\gamma_2"(ro_g))_A &\Leftrightarrow X_1 \in [y_1]_{("\gamma_1"(ro_g) \cap "\gamma_2"(ro_g))} \subseteq A \\
&\Leftrightarrow X_1 \in \{["\gamma_1"(uni_1) \cap "\gamma_2"(uni_1)] \cup [\overline{"\gamma_1"(uni_1)} \cap \overline{"\gamma_2"(uni_1)}] \cup [\underline{"\gamma_1"(uni_1)} \cap \underline{"\gamma_2"(uni_1)}]\} \\
&\subseteq A \\
&\Leftrightarrow X_1 \in \{["\gamma_1"(uni_1) \cap "\gamma_2"(uni_1)] \subseteq A\} \cap \{[\overline{"\gamma_1"(uni_1)} \cap \overline{"\gamma_2"(uni_1)}] \subseteq A\} \\
&\cap \{[\underline{"\gamma_2"(uni_1)} \cap \underline{"\gamma_2"(uni_1)}] \subseteq A\} \\
&\Leftrightarrow X_1 \in \{["\gamma_1"(uni_1) \cup \overline{"\gamma_1"(uni_1)} \cup \underline{"\gamma_1"(uni_1)}] \subseteq A\} \cap \{["\gamma_2"(uni_1) \\
&\cup \overline{"\gamma_2"(uni_1)} \cup \underline{"\gamma_2"(uni_1)}] \subseteq A\} \\
&\Leftrightarrow X_1 \in ("\gamma_1"(ro_g))_A \cap ("\gamma_2"(ro_g))_A
\end{aligned}$$

*Hence the lower approximation of any set with respect to intersection of .rough group equivalence relation is equal to intersection of lower approximation of the set.*

## 5 Results and Discussion

To develop rough set theory, it is essential to look at its algebraic structure. By considering upper and lower approximations of rough sets, we defined rough groups in a more extended sense than previous approaches such as considering the upper approximation of any subset in a finite universe and demonstrating closure, associativity, and identity in the upper approximation, but the inverse exists in the set itself [23] also considering abstract groups as universe set and its normal subgroups as equivalence relation [9]. An exploration of the expansive properties of rough groups based on the rough group equivalence relation has been presented in this paper.

## 6 Conclusions

The algebraic aspects of rough set theory have been integral to the development of rough set theory concept as algebraic structures allow the detailed study of set theoretic properties. Rough groups are introduced in this paper using both upper and lower approximations to rough sets within a finite universe. Further more, rough groups have been shown to have expansive proper ties such as upper approximation of intersection of rough groups is contained in its intersection while lower approximation of inter section of rough groups is equal to its intersection. Also upper approximation of union of rough groups is equal to it union. Based on rough group equivalence, we have defined a rough group approximation space and derived the upper and lower approximations of any set. More studies will be conducted in the future to examine rough group properties in greater detail. A similar extension can be made to other algebraic structures as well.

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# Neutrosophic Generalized Pareto Distribution

**Nahed I. Eassa1,*, Hegazy M. Zaher1, Noura A. T. Abu El-Magd2**
1Faculty of Graduate Studies for Statistical Research, Cairo University, Egypt
2Faculty of Politics and Economics, Beni-Suef University, Egypt

## A B S T R A C T

The purpose of this paper is to present a neutrosophic form of the generalized Pareto distribution (NGPD) which is more flexible than the existing classical distribution and deals with indeterminate, incomplete and imprecise data in a flexible manner. In addition to this, NGPD will be obtained as a generalization of the neutrousophic Pareto distribution. Also, the paper introduces its special cases as neutrosophic Lomax distribution. The mathematical properties of the proposed distributions, such as mean, variance and moment generating function are derived. Additionally, the analysis of reliability properties, including survival and hazard rate functions, is mentioned. Furthermore, neutrosophic random variable for Pareto distribution was presented and recommended using it when data in the interval form follow a Pareto distribution and have some sort of indeterminacy. This research deals the statistical problems that have inaccurate and vague data. The proposed model NGPD is widely used in finance to model low probability events. So, it is applied to a real-world data set to modelling the public debt in Egypt for the purpose of dealing with neutrosophic scale and shape parameters, finally the conclusions are discussed.

**Keywords** Lomax Distribution, Neutrosophic Logic , Generalized Pareto Distributionwith Tow Parameters, Neutrosophic Pareto Distribution, Neutrosophic Exponential Distribution, Neutrosophic Uniform

## 1. Introduction

The Generalized Pareto distribution (GPD) is widely used in extreme value theory, engineering, industrial and finance. GPD is related to several distributions such as Exponential, Lomax, and Uniform distributions. To add flexibility to this model, neutrosophic Logic has been used. Neutrosophy was presented by Smarandache in 1995, as a generalization for the fuzzy logic and intuitionist fuzzy logic [1]. Fuzzy logic which is the special case of neutrosophic logic gives information only about the measures of truth and falseness. The neutrosophic logic gives information about the measure of indeterminacy additionally. The neutrosophic logic used the set analysis, where any type of set can be used to capture the data inside the intervals.

Neutrosophic statistics which utilize the idea of neutrosophic logic are found to be more efficient than classical statistics [1]. Neutrosophic statistics deal with the data having imprecise, interval, and uncertain observations. Neutrosophic statistics reduce to classical statistics when no indeterminacy is found in the data or the parameters of statistical distribution. Various applications of neutrosophic logic can be read in [2, 3].

Many researchers have introduced neutrosophic logic as an extended and generalized approach to the classical distributions such as neutrosophic Weibull distribution [4] and its several families, neutrosophic binomial distribution and neutrosophic normal distribution [5], neutrosophic multinomial distribution, neutrosophic Poisson [6], neutrosophic exponential [7,8], neutrosophic distribution, neutrosophic gamma distribution [9], neutrosophic beta distribution [10], and neutrosophic Rayleigh model [11]. The neutrosophic Pareto distribution (NPD), generalization of the Pareto distribution is developed by Zahid khan et al. [12]. Almarashet and aslam [13] presented a repetitive sampling control chart for the gamma distribution under the indeterminate environment.

This paper proposes neutrosophic Pareto distribution with neutrosophic random variables, Neutrosophic distribution and neutrosophic Generalized Pareto distributions.

The paper is organized as follows: The next section describes the neutrosophic generalized Pareto with two parameters and some special cases. In section 3, it studies the probability density function (pdf), cumulative density function (cdf), and hazard rate function of the neutrosophic Lomax distribution model. The mathematical statistics studied in the subsequent section such as mean and variance.

Then, we will introduce the neutrosophic Pareto distribution model in section 4. Finally, section 5 concludes the research outcomes.

## 2. The Neutrosophic Generalized Pareto Distribution (NGPD)

The neutrosophic generalized Pareto distribution is a family of continuous probability distributions. It is often used to model the tails of another distribution. NGPD will be obtained as a generalization of the neutrousophic Pareto distribution given by Zahed Khan et al. [12]. The NGPD is related to several distributions such as neutrosophic Exponential, Uniform distribution which is introduced by Carlos Granadosa et al. [14] and neutrosophic Lomax distribution, as it will be shown below.

We can have defined the NGPD with neutrosophic parameters as follows:

- The probability density function (*pdf*) is given by:

$$f(x,\alpha_N,\beta_N) = \frac{1}{\beta_N}(1+\frac{\alpha_N x}{\beta_N})^{-\frac{1}{\alpha_N}-1} \; ; x > 0 \, , \alpha_N, \beta_N > 0 \quad (1)$$

when shape parameter $\alpha_N = 0$, the density is:

$$f(x,0,\beta_N) = \frac{1}{\beta_N} e^{-(x)/\beta_N} \; ; x > 0 \; , \beta_N > 0 \quad (2)$$

(Exponential distribution)

- The cumulative distribution function (cdf):

$$F(x,\alpha_N,\beta_N) = 1 - \left(1+\frac{\alpha_N x}{\beta_N}\right)^{\frac{-1}{\alpha_N}}; \quad \alpha_N \neq 0, \beta_N > 0 \quad (3)$$

$$F(x,\alpha_N = 0,\beta_N) = 1 - e^{-\frac{x}{\beta_N}}; \; \alpha_N = 0, \beta_N > 0$$

$$x \geq 0 \, , when \alpha_N \geq 0 \; and \; 0 \leq x \leq -\frac{\beta_N}{\alpha_N} when \alpha_N < 0 \quad (4)$$

- The hazard rate function:

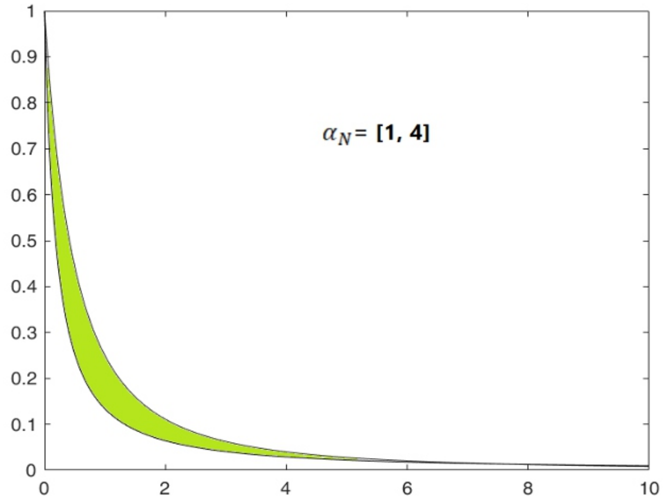
$$h(x,\alpha_N,\beta_N) = (\beta_N x + \alpha_N)^{-1}; \; x > 0, \alpha_N, \; \beta_N > 0 \quad (5)$$

- The survival function:

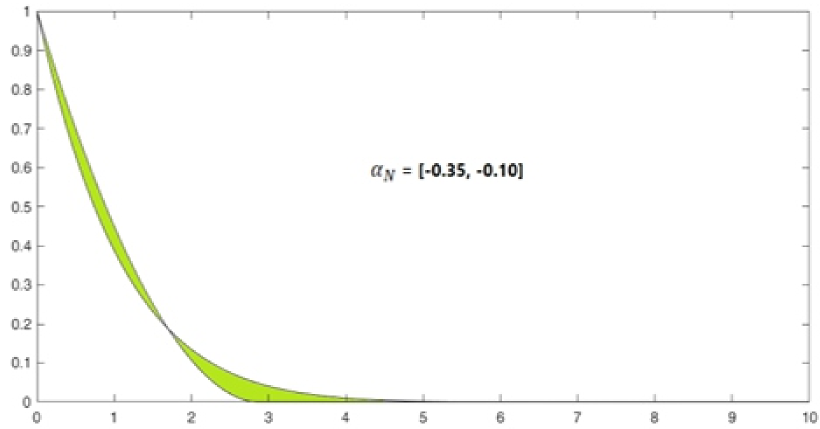
$$s(x,\alpha_N,\beta_N) = \left(\frac{\alpha_N}{\beta_N x+\alpha_N}\right)^{\frac{1}{\beta_N}}; x > 0 \; and \alpha_N, \beta_N > 0 \quad (6)$$

Where $\alpha_N$ is neutrosophic scale parameter and $\beta_N$ is a neutrosophic shape parameter $\alpha_N \in (\alpha_L, \alpha_U)$ and $\beta_N \in (\beta_L, \beta_U)$.

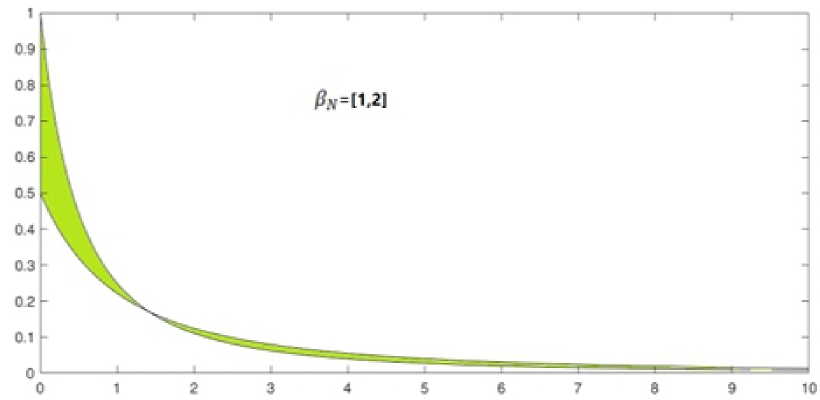
The pdf curves of the NGPD are presented in Figure 1. The graphical expression of the $f(x,\alpha_N,\beta_N)$ of the neutrosophic generalized Pareto with imprecise parameters.




(A) The pdf of the NGPD with neutrosophic Shape parameters




(B) The pdf of the NGPD with neutrosophic shape parameters $\alpha_N < 0$




(C) The pdf of the NGPD with neutrosophic Scale parameters

**Figure 1.** Theneutrosophic shape parameter and the neutrosophic scale parameter

Figure 1-A shows the pdf curve of the distribution with the neutrosophic shape parameter $\alpha_N$ =[1, 4] if the data are believed to be NGPD with $\beta_N = 1$. Figure 1-B shows the neutrosophic shape parameter if $\alpha_N < 0$. Also, Figure 1-C shows the NGPD with neutrosophic scale parameters $\beta_N$ = [1, 2] and $\alpha_N = 1$.

### 2.2. The Properties of NGPD

We will introduce some properties of NGPD, such as mean, variance and special cases.

**1- The Mean of NGPD**

$$\mu_N = E(x) = \int_0^\infty x \frac{1}{\beta_N}(1+\frac{\alpha_N x}{\beta_N})^{-\frac{1}{\alpha_N}-1} dx$$

$$= \frac{1}{\beta_N}\int_0^\infty x(1+\frac{\alpha_N x}{\beta_N})^{-\frac{1}{\alpha_N}-1} dx$$

$$\mu_N = \frac{\beta_N}{1-\alpha_N} \quad ; \quad \alpha_N < 1$$

$$\mu_N = (\frac{\beta_l}{1-\alpha_l}, \frac{\beta_u}{1-\alpha_u}) \tag{7}$$

**2- The Variance:**

$$V(x) = E(x^2) - (\mu_N)^2$$

$$E(x^2) = \int_0^\infty x^2(1+\frac{\alpha_N x}{\beta_N})^{-\frac{1}{\alpha_N}-1} dx$$

$$= \frac{1}{1-2\alpha_N} \quad ; \quad \alpha_N < \frac{1}{2}$$

$$var(x) = \frac{{\beta_N}^2}{(1-2\alpha_N)(1-\alpha_N)^2} \quad ; \alpha_N < \frac{1}{2} \tag{8}$$

**3- Special Cases:**

1) If the neutrosophic shape parameter $\alpha_N = 0$ in equation 1, the NGPD will be equivalent to neutrosophic exponential distribution as shown in equation 2.
2) If $\alpha_N > 0$, the NGPD will be neutrosophic Lomax distribution as shown in equation (9).
3) If $\alpha_N = -1$, then the NGPD will be neutrosophic uniform (0,$\beta_N$).

$$f(x, -1, \beta_N) = \frac{1}{\beta_N}(1+\frac{-x}{\beta_N})^{1-1} = \frac{1}{\beta_N} \approx Uniform(0, \beta_N)$$

## 3. The Neutrosophic Lomax Distribution (NLD)

The Lomax distribution, also called the Pareto Type II distribution was presented by Lomax in the mid of the last century, and it is a heavy-tail probability distribution used in economics and business failure data. To improve the flexibility of the existing model, there are some methods as increasing the number of parameters, making some transformation [15] and proper mixing of two distributions [16]. In this paper, we present a new model (neutrosophic logic) to add more flexibility with incomplete data and indeterminacy data.

The probability density function of a Lomax distribution with neutrosophic shape parameter $\alpha_N > 0$ and neutrosophic scale parameter $\beta_N > 0$ is given by

$$f_N(x) = \frac{\alpha_N}{\beta_N}\left(1+\frac{x}{\beta_N}\right)^{-(\alpha_N+1)} ; x \geq 0 \; and \beta_N, \alpha_N > 0 \tag{9}$$

where $\alpha_N = (\alpha_L, \alpha_U)$ and $\beta_N = (\beta_L, \beta_U)$. Note that the NGPD differs from the classical distribution, when the indeterminate part with shape and scale parameters is considered zero in the neutrosophic Lomax distribution, that is, $\alpha_L = \alpha_U = \alpha$ and $\beta_L = \beta_U = \beta$, it tends to classical Lomax distribution.

### 3.1. The Properties of the Neutrosophic Lomax Distribution (NLD)

**6- The Inverse Distribution Function of the NLD:**

$$F^{-1}(u) = \frac{(1-u)^{-1/\beta_N}\frac{-1}{\alpha_N}}{\alpha_N} \tag{15}$$

## 4. Neutrosophic Pareto Distribution (NPD) Model

The Pareto distribution, is the power-law probability distribution that is used in description of social, quality control, scientific, geophysical, actuarial and many other areas. A neutrosophic Pareto distribution is a classical Pareto distribution but its parameters nor its variables are unclear or imprecise. Zahed Khan et al. [12] introduced the neutrosophic Pareto distribution model in the neutrosophic parameter and studied its properties. In this paper we will investigate the neutrosophic random variable for Pareto distribution.

The neutrosophic random variable x follows the NPD model with the following neutrosophic density function:

$$f_N(x) = \frac{\alpha_N \beta_N^{\alpha_N}}{x_N^{\alpha_N+1}}(1+I_N)\; ; \; for x_N > \beta_N and \beta_N, \alpha_N > 0 \tag{16}$$

where $\alpha_N$ = ( $\alpha_L, \alpha_U$) is neutrosophic shape parameter, $\beta_N = (\beta_L, \beta_U)$ is neutrosophic scale parameter and $x_N \in (x_L, x_U)$ is neutrosophic random variable, where $x_N = x_L + x_U I_N$ and $I_N \in (I_L, I_U)$ is an indeterminacy interval where N is the neutrosophic statistical number. The netrosophic Pareto distribution tends to the classical distribution when $I_N = 0$.

The corresponding neutrosophic cumulative distribution is:

We will introduce here the properties of the neutrosophic Lomax distribution (NLD), such as the cumulative distribution function, the mean, variance, the survivor function, the hazard function and the inverse distribution function.

**1- The Cumulative Distribution Function (cdf) of the NLD:**

$$F_N(x) = 1 - \left(1 + \frac{x}{\beta_N}\right)^{-\alpha_N}; x \geq 0 \; and \beta_N, \alpha_N > 0 \quad (10)$$

**2- The Mean of the NLD:**

$$E(x_N) = \frac{\beta_N}{\alpha_N - 1} for \alpha_N > 1 \quad (11)$$

**3- The Variance of the NLD:**

$$v(x_N) = \frac{(\beta_N)^2 \alpha_N}{(\alpha_N - 1)^2(\alpha_N - 2)} for \alpha_N > 2 \quad (12)$$

**4- The Survivor Function of the NLD:**

$$S(x) = (1 + \beta_N x)^{-\alpha_N} \quad (13)$$

**5- The HAzard Function of the NLD:**

$$H(x) = \alpha_N \ln(1 + \beta_N x) \quad (14)$$

$$V(x_N) = E(x_N^2) - \mu^2$$

$$E(x_N^2) = (1 + I_N)\int_{\beta_N}^{\infty} (x_N)^2 \frac{\alpha_N \beta_N^{\alpha_N}}{x_N^{\alpha_N + 1}} d\,x_N$$

$$= \frac{\alpha_N \beta_N^2}{\alpha_N - 2}(1 + I_N) for \alpha_N > 2$$

$$V(x_N) = \frac{\alpha_N \beta_N^2}{\alpha_N - 2}(1 + I_N) - \left[\left(\frac{\alpha_N \beta_N}{\alpha_N - 1}\right)(1 + I_N)\right]^2$$

$$for \alpha_N > 2 \quad (19)$$

**3- The Moment Generating Function of the NPD:**

$$\mu_{x_N}(t) = E(e^{tx_N}) = (1 + I_N)\int_{\beta_N}^{\infty} e^{tx_N} \frac{\alpha_N \beta_N^{\alpha_N}}{x_N^{\alpha_N + 1}} d\,x_N$$

$$= (1 + I_N)\alpha_N(-\beta_N t)^{-\alpha_N} \quad (20)$$

To add more flexibility to the neutrosophic Pareto distribution, various families and generalization of the neutrosophic distribution have been derived including the neutrosophic generalized Pareto distribution with tow parameter.

$$s(x, \alpha_N, \beta_N) = \left(\frac{\alpha_N}{\beta_N x + \alpha_N}\right)^{\frac{1}{\beta_N}}; \; x > 0 \; and \alpha_N, \beta_N > 0 \quad (21)$$

$$F_N(x) = \left(1 - \left(\frac{\beta_N}{x_N}\right)^{\alpha_N}\right)(1 + I_N), \; x_N > \beta_N and \beta_N, \alpha_N > 0 \quad (17)$$

### 4.1. Statistical Properties of Neutrosophic Pareto Distribution

The main properties of mathematical statistics like mean, variance, quantile and moment generating functions have been studied in this section.

**1- The Mean of NPD:**

$$E(x_N) = (1 + I_N)\int_{\beta_N}^{\infty} x_N \frac{\alpha_N \beta_N^{\alpha_N}}{x_N^{\alpha_N + 1}} d\,x_N$$

$$= (1 + I_N)\alpha_N \beta_N^{\alpha_N}\int_{\beta_N}^{\infty} x_N^{-\alpha_N} dx_N$$

$$= \frac{\alpha_N \beta_N}{\alpha_N - 1}(1 + I_N) \quad \text{for } \alpha_N > 1$$

$$\mu_N = \left[\frac{\alpha_l \beta_l}{\alpha_l - 1}(1 + I_l), \frac{\alpha_u \beta_u}{\alpha_u - 1}(1 + I_u)\right] \quad (18)$$

**2- The Variance of the NPD:**

where $\alpha_N$ is a neutrosophic scale parameter and $\beta_N$ is a neutrosophic shape parameter. $\alpha_N \in (\alpha_L, \alpha_U)$ and $\beta_N \in (\beta_L, \beta_U)$.

## 5. Real Application

In this part, a practical application using a real-world data set has been used to assess the interest in the NGPD model.

The public debt in Egypt is increasing at an alarming rate This study has applied the extreme value theory in modelling the public debt where NGPD has been used. In Figure 2, the PDF-plot demonstrates that the GPD will fit public debt in Egypt.

The data under consideration includes a set of public debt in Egypt covering the period from 2000 to 2022. The data reported by central bank of Egypt (https://www.cbe.org.eg/ar/economic-research) are shown in Table 1. The observations in this dataset represent the public debt (in millions dollar). The data from sources are crisp values; for the purpose of illustration we treat the data set such as shown in Table 1 and the graphical summary of crisp data is shown in Figure 2.

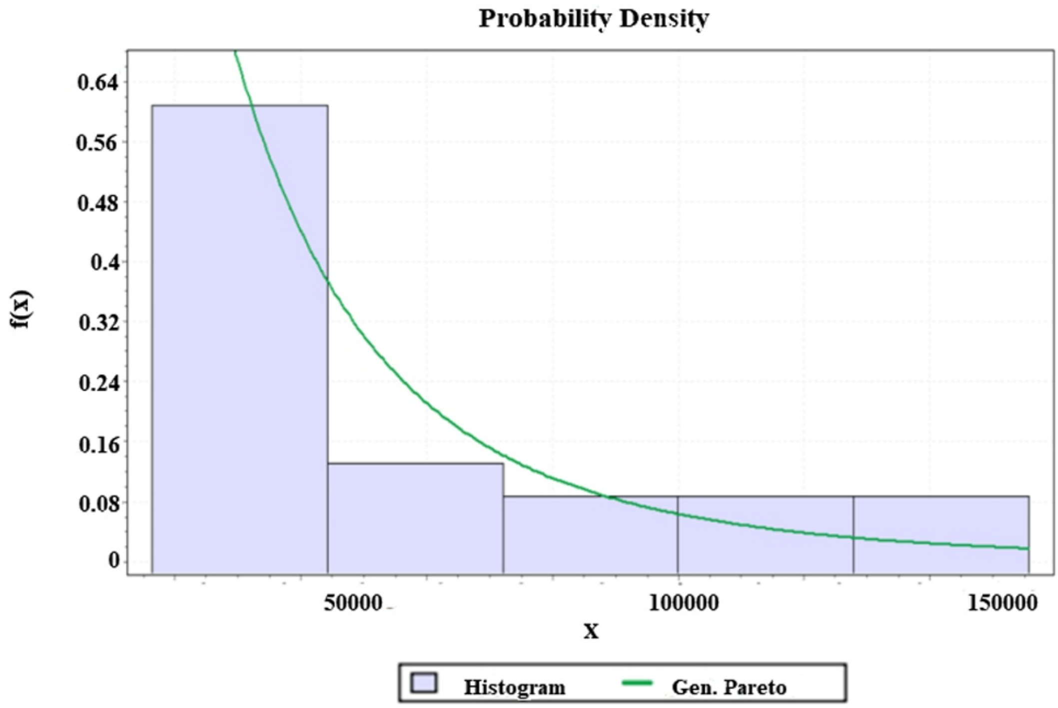



**Figure 2.** The graphical summary of crisp data

**Table 1.** Sample from data of Public debit in Egypt

| | | | |
|---|---|---|---|
| [27780.1, 27783.3] | 26560 | 28660 | 29396.2 |
| 16384.8 | 29592.6 | 29898 | |
| [33850 ,33892] | 31531.1 | [46060,46067.1] | 29871.8 |
| 33694.2 | 34905.7 | 34384.5 | 43233.4 |
| 48062.9 | 55764.4 | [79032.8,79033] | [92613.9, 92643.9] |
| 108699.1 | 123490.5 | 137859.6 | [155608.9,155708.9] |

The data aren't precisely reported but are provided in intervals. These uncertainties in the sample render the classical generalized Pareto models inapplicable. On the contrary, the NGP may effectively be used to investigate the properties of the neutrosophic data set. The descriptive statistics of the public debt data using the NGP model are given in Table 2. The following Table 3 is the result of fitting NGPD:

**Table 2.** Parameters Estimation

| parameters | estimate |
|---|---|
| Shape parameter $\alpha_N$ | [0.25397,0.2536] |
| Scale parameter $\beta_N$ | [26528,26543] |
| threshold | [19968,19955] |

**Table 3.** Neutrosophic statistics of public debt data by using NGP

| Descriptive measures | |
|---|---|
| Mean | [55516,55527] |
| Variance | [2.5662E+9,2.5697E+09] |
| Mode | [19955,19968] |
| Skewness | [7.3581,7.3890] |

generalized Pareto distribution. To test the assumption, we applied the neutrosophic Kolmogrov-Smirnov (NK-S) test, which are the generalization of the Kolmogrov-Smirnov [17]. The results are shown in Table 4.

The neutrosophic null hypothesis that the sample coming from the NGPD is accepted when $D_N \in [D_L, D_U] < D_{\alpha,N}$.where $D_{\alpha,N}$ is a neutrosophic critical value. Note here that the $D_N \in [0.15589,\ 0.15601] < 0.28358$ then the data follow the NGPD.

## 6. Conclusions

The NGPD plays an important role in modelling extreme value when data is incomplete or indeterminate. This paper discussed the new neutrosophic distribution (neutrosophic Generalized Pareto distribution). The neutrosophic Lomax distribution and neutrosophic Pareto distribution are a special case study from the NGPD. Properties of mathematical characteristics of the proposed distributions under an indeterminacy environment are described. The effectiveness of the proposed model NGPD has been demonstrated by using a real dataset on Public debit data.

In the future, the estimation of parameters for the NGPD

Table 2 and Table 3 show the estimated neutrosophic measures based on the NGPD. All the estimated values are expressed as intervals because of indeterminacies inherent in the analyzed dataset.

To estimate the Generalized Pareto Distribution (GPD) model parameters, we find that the shape parameter $\alpha_N$= [0.25397,0.2536], the scale parameter, $\beta_N$=[26528,26543] and the Threshold is [19968, 19955] as shown in Table 2.

### 5.1. Goodness of Fit Test for Neutrosophic Data

**Table 4.** Goodness-of fit test for the data by NK-S tests

| Model | NK-S Critical value $D_{0.05,23}$ | $\alpha$=0.05 Statistic($D_N$) |
|---|---|---|
| NGPD | 0.28358 | [0.15589, 0.15**601**] |

For the public debt in Egyptdata, we interested in testing the assumption that the data follows a neutrosouphic

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can be studied and also other neutrosophic families can be presented.

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# Resolution of Linear Systems Using Interval Arithmetic and Cholesky Decomposition

**Benhari Mohamed amine, Kaicer Mohammed**
Laboratory of Analysis, Geometry and Applications, Faculty of Sciences, Ibn-Tofail University, Morocco

## A B S T R A C T

This article presents an innovative approach to solving linear systems with interval coefficients efficiently. The use of intervals allows the uncertainty and measurement errors inherent in many practical applications to be considered. We focus on the solution algorithm based on the Cholesky decomposition applied to positive symmetric matrices and illustrate its efficiency by applying it to the Leontief economic model. First, we use Sylvester's criterion to check whether a symmetric matrix is positive, which is an essential condition for the Cholesky decomposition to be applicable. It guarantees the validity of our solution algorithm and avoids undesirable errors. Using theoretical analyses and numerical simulations, we show that our algorithm based on the Cholesky decomposition performs remarkably well in terms of accuracy. To evaluate our method in concrete terms, we apply it to the Leontief economic model. This model is widely used to analyze the economic interdependencies between different sectors of an economy. By considering the uncertainty in the coefficients, our approach offers a more realistic and reliable solution to the Leontief model. The results obtained demonstrate the relevance and effectiveness of our algorithm for solving linear systems with interval coefficients, as well as its successful application to the Leontief model. These advances are crucial for fields such as economics, engineering, and the social sciences, where data uncertainty can greatly affect the results of analyses. In summary, this article highlights the importance of interval arithmetic and Cholesky's method in solving linear systems with interval coefficients. Applying these tools to the Leontief model can help you better understand the impact of uncertainty and make informed decisions in a variety of fields, including economics and engineering.

**Keywords** Arithmetic Interval, Interval Matrix, System of Interval Linear Equations, Decomposition of Cholesky

## 1. Introduction

Interval arithmetic is the branch of mathematics concerned with the properties and operations of numerical intervals. Although it may seem abstract at first glance, interval arithmetic has practical applications in many fields, from computing and engineering to the physics and economics. It allows the manipulation of intervals rather than exact numbers. This mathematical discipline allows uncertainty, imprecision, or measurement error to be expressed formally and rigorously. In addition, interval arithmetic offers a new approach to complex mathematical problems where the quantities involved are uncertain or difficult to evaluate accurately, which can lead to a deeper understanding of the concepts and the exploration of innovative solutions.

This article focuses on solving the AX = B system applied to matrices with interval coefficients using the Cholesky decomposition, a method used to factor a symmetric positive definite matrix into a product of two lower triangular matrices and its transpose with interval coefficients. More precisely, this factorization makes it possible to solve the system more efficiently.

Using this approach in the Leontief model, the relationships between the different economic sectors are generally represented by a system of linear equations [1] which describe the total demand of each sector as a function of the total production of the other sectors. However, there are often uncertainties and errors in the data used to construct these equations. Where interval arithmetic is used, uncertain or poorly measured values can be represented by intervals instead of precise numbers. This allows uncertainties in the results to be considered and the impact of errors on economic predictions to be quantified.

In this article, we will discuss the Cholesky decomposition to solve a linear system with interval coefficients and apply this decomposition to an economic model called the Leontief model.

# 2 Interval arithmetic

## 2.1 Elementary operations on intervals

Let $I\mathbb{R} = \{\widehat{a} = [a_1; a_2] : a_1 \leq a_2 and a_1, a_2 \in \mathbb{R}\}$ be the set of all proper intervals and $\overline{I\mathbb{R}} = \{\widehat{a} = [a_1; a_2] : a_1 > a_2; a_1, a_2 \in \mathbb{R}\}$ be the set of all improper intervals on the real line R. If $a_1 = a_2 = a$, then $\widehat{a} = [a, a] = a$ is a real number (or a degenerate interval). We shall use the terms "interval" and "interval number" interchangeably. The mid-point and width(or half-width) of an interval number $\widehat{a} = [a_1, a_2]$ are defined as $m(\widehat{a}) = \frac{a_1 + a_2}{2}$ and $w(\widehat{a}) = \frac{a_2 - a_1}{2}$. We denote the set of generalized intervals(proper and improper) by :

$$K\mathbb{R} = I\mathbb{R} \cup \overline{I\mathbb{R}} = \{[a_1; a_2] : a_1, a_2 \in \mathbb{R}\}$$

The set of generalized intervals $K\mathbb{R}$ is a group with respect to addition and multiplication operations of zero free intervals, while maintaining the inclusion monotonicity.

The "dual" is an important monadic operator proposed by **kaucher** that reverses the end-points of the intervals in $K\mathbb{R}$. For $\widehat{a} = [a_1, a_2] \in K\mathbb{R}$, its dual is defined by $dual(\widehat{a}) = dual([a_1, a_2]) = [a_2, a_1]$. The opposite of an interval $\widehat{a} = [a_1, a_2]$ is $opp([a_1, a_2]) = [-a_1, -a_2]$ which is the additive inverse of $[a_1, a_2]$ and $\left[\frac{1}{a_1}, \frac{1}{a_2}\right]$ is the multiplicative inverse of $[a_1, a_2]$, provided $0 \notin [a_1, a_2]$.

That is, $\widehat{a} + (-dual(\widehat{a})) = [0, 0]$ and $\widehat{a} \times \frac{1}{dual(\widehat{a})} = [1, 1]$.

**Ganesan and Veeramani** [2] proposed new interval arithmetic on $\mathbb{IR}$. We extend these arithmetic operations to the set of generalized interval numbers $K\mathbb{R}$ and incorporate the

## 2.2 Interval matrices

### 2.2.1 Definitions

A square interval matrix $\widehat{A_{n,n}}$ of order n is defined as a matrix and can be written in the form [3] :

$$\widehat{A_{n,n}} = (\widehat{a_{i,j}})_{1 \leq i \leq n, 1 \leq j \leq n} = \begin{pmatrix} \widehat{a_{1,1}} & \widehat{a_{1,2}} & \cdots & \widehat{a_{1,n}} \\ \widehat{a_{2,1}} & \widehat{a_{2,2}} & \cdots & \widehat{a_{2,n}} \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{a_{n,1}} & \widehat{a_{n,2}} & \cdots & \widehat{a_{n,n}} \end{pmatrix}$$

If $\widehat{A_{n,n}}$ and $\widehat{B_{n,n}}$ are interval matrices and $\alpha \in \mathbb{R}$, then :

- $\alpha \widehat{A_{n,n}} = \alpha(\widehat{a_{i,j}})_{1 \leq i \leq n, 1 \leq j \leq n}$
- $\widehat{A_{n,n}} + \widehat{B_{n,n}} = (\widehat{a_{i,j}} + \widehat{b_{i,j}})_{1 \leq i \leq n, 1 \leq j \leq n}$
- $\widehat{A_{n,n}} - \widehat{B_{n,n}} = (\widehat{a_{i,j}} - \widehat{b_{i,j}})_{1 \leq i \leq n, 1 \leq j \leq n}$ if $\widehat{A} \neq \widehat{B}$, and $\widehat{A_{n,n}} - \widehat{B_{n,n}}$=[0; 0] if $\widehat{A} = \widehat{B}$
- $\widehat{A_{n,n}}.\widehat{B_{n,n}} = \left(\sum_{k=1}^{n} \widehat{a_{i,k}}.\widehat{b_{k,j}}\right)_{1 \leq i \leq n, 1 \leq j \leq n}$

**The transpose of a square matrix** $\widehat{A_{n,n}}$ of order n is an interval coefficient matrix denoted by: $\widehat{A_{n,n}}^T$, obtained by exchanging the rows and columns of $\widehat{A_{n,n}}$ .

**A symmetric matrix with interval coefficient**: A symmetric matrix $\widehat{A_{n,n}}$ of order n is a square matrix which is equal to its own transpose, i.e. such that $\widehat{a_{i,j}} = \widehat{a_{j,i}}$ for all $i$ and $j$ between 1 and n, where the $\widehat{a_{i,j}}$ are the interval matrix coefficients and n is its order.

### 2.2.2 Determinant of an interval coefficient matrix

For any square matrix $\widehat{A_{n,n}}$ of order n with interval coefficient corresponds a value called the determinant of $\widehat{A_{n,n}}$ noted $det(\widehat{A_{n,n}})$, the method of calculating the determinant remains the same from the case of matrices with interval coefficients except the determinant of un interval matrix in an interval [3].

It is easy to see that most of the properties of the determinant of a classical matrix are valid for the determinant of the interval matrix.

concept of dual.
For $\widehat{a} = [a_1; a_2]$ , $\widehat{b} = [b_1; b_2] \in K\mathbb{R}$ and for $* \in \{+; -; \times; \div\}$ we define :

$$\widehat{a} * \widehat{b} = [m(\widehat{a}) * m(\widehat{b}) - k; m(\widehat{a}) * m(\widehat{b}) + k] \text{ and}$$
$$k = min\{(m(\widehat{a}) * m(\widehat{b}) - \alpha; \beta - (m(\widehat{a}) * m(\widehat{b}))\}$$

$\alpha$ and $\beta$ are the end points of the interval $\widehat{a}$ and $\widehat{b}$

If $\widehat{a} = [a_1; a_2] \in K\mathbb{R}$ is positive, we define $\sqrt{\widehat{a}}$ as

$$\sqrt{\widehat{a}} = [\sqrt{a_1}; \sqrt{a_2}]$$

It is clear that by this notation, we have $\sqrt{\widehat{a}} \times \sqrt{\widehat{a}} = \widehat{a}$

#### 3.1.2 Sylvester's criterion :

For a symmetric matrix with interval coefficient $\widehat{A_{n,n}}$ of size n to be positive definite, it is necessary and sufficient that the n principal minors $(A_p)_{1 \leqslant p \leqslant n}$ are strictly positive intervals.

#### 3.1.3 Example 1:

Consider the symmetric interval matrix

$$\widehat{A} = \begin{pmatrix} [3.7; 4.3] & [-1.5; -0.5] & [0; 0] \\ [-1.5; -0.5] & [3.7; 4.3] & [-1.5; -0.5] \\ [0; 0] & [-1.5; -0.5] & [3.7; 4.3] \end{pmatrix}$$ Let's check if $\widehat{A}$ is a positive definite square symmetric matrix using **Sylvester**'s criterion:
We have :

$|[3.7; 4.3]| = [3.7; 4.3] > 0$
and $\begin{vmatrix} [3.7; 4.3] & [-1.5; -0.5] \\ [-1.5; -0.5] & [3.7; 4.3] \end{vmatrix} = [11.94; 18.06] > 0$
and
$$\begin{vmatrix} [3.7; 4.3] & [-1.5; -0.5] & [0; 0] \\ [-1.5; -0.5] & [3.7; 4.3] & [-1.5; -0.5] \\ [0; 0] & [-1.5; -0.5] & [3.7; 4.3] \end{vmatrix} = [37.10; 74.89] > 0$$

So $\widehat{A}$ is symmetric positive definite.

### 3.2 Cholesky decomposition

If A is a square matrix with interval coefficient symmetric and positive definite, then there exists a lower triangular matrix with interval coefficient F which satisfies:

$$A = F.F^T$$

This decomposition, called the factorization of **Cholesky**, is the product of a lower triangular matrix $F$ by its transpose.
Let $\widehat{A_{n,n}}$ be a square matrix of order n and interval coefficient such that:

$$\widehat{A_{n,n}} = \begin{pmatrix} \widehat{a_{1,1}} & \widehat{a_{1,2}} & \cdots & \widehat{a_{1,n}} \\ \widehat{a_{2,1}} & \widehat{a_{2,2}} & \cdots & \widehat{a_{2,n}} \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{a_{n,1}} & \widehat{a_{n,2}} & \cdots & \widehat{a_{n,n}} \end{pmatrix}$$

## 3 Solving the $\widehat{A}$ $\widehat{X}$=$\widehat{B}$ system using the Choleski decomposition

### 3.1 Positive definite matrix and the Sylvester criterion

#### 3.1.1 Definitions

Let $\widehat{A_{n,n}}$ be a symmetric square matrix with interval coefficient of size n . We call principal minors the determinants of the n matrices $A_p = (\widehat{a_{ij}})$, for p ranging from 1 to n.
Sylvester's criterion provides a simple method for testing the positive definiteness of a matrix $\widehat{A_{n,n}}$.

With $i = j+1, j+2, ...., n$
And

$$\widehat{b_{j,j}} = \sqrt{\widehat{a_{j,j}} - \sum_{k=1}^{j-1} (\widehat{b_{j,k}})^2} \quad (2)$$

With $j = 1, 2, ...., n$

**Consequences**

If A is a square matrix of order 3 and satisfies all the conditions, then from (1) and (2) the interval coefficients of the matrix F are defined by:

$$\widehat{b_{1,1}} = \sqrt{\widehat{a_{1,1}}}$$
$$\widehat{b_{2,1}} = \frac{\widehat{a_{2,1}}}{\widehat{b_{1,1}}}$$
$$\widehat{b_{2,2}} = \sqrt{\widehat{a_{2,2}} - (\widehat{b_{2,1}})^2}$$
$$\widehat{b_{3,1}} = \frac{\widehat{a_{3,1}}}{\widehat{b_{1,1}}}$$
$$\widehat{b_{3,2}} = \frac{\widehat{a_{3,2}} - \widehat{b_{3,1}} \times \widehat{b_{2,1}}}{\widehat{b_{2,2}}}$$
$$\widehat{b_{3.3}} = \sqrt{\widehat{a_{3.3}} - (\widehat{b_{3.1}})^2 - (\widehat{b_{3.2}})^2}$$

**Solving the $\widehat{A}$ $\widehat{X}$=$\widehat{B}$ system**

To solve a linear system involving interval matrices, we seek to find the smallest interval vector containing the set of vectors $\widehat{X}$ such that there exists a point matrix $A \in \widehat{A}$ and $B \in \widehat{B}$ and we have the equality $Ax = B$.

**Calculation algorithm**

Let $\widehat{A}$ and $\widehat{B}$ be two square matrices of order $n$ with interval coefficients. Solving the system $\widehat{A}\widehat{X} = \widehat{B}$ consists of going through the following steps:
**Step 1** : Check if $\widehat{A}$ is a positive definite symmetric matrix using Sylvester's criterion.
**Step 2** : Decompose $\widehat{A}$ as $\widehat{F} \times \widehat{F}^T$ using Cholesky decomposition.

If $F$ a lower triangular Matrix with interval coefficient that satisfies $A = F.F^T$:
So :

$$\widehat{F_{n,n}} = \begin{pmatrix} \widehat{b_{1,1}} & \widehat{0} & \cdots & \widehat{0} \\ \widehat{b_{2,1}} & \widehat{b_{2,2}} & \cdots & \widehat{0} \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{b_{n,1}} & \widehat{b_{n,2}} & \cdots & \widehat{b_{n,n}} \end{pmatrix}$$

Such as :

$$\widehat{b_{i,j}} = \frac{1}{\widehat{b_{j,j}}} \left( \widehat{a_{j,j}} - \sum_{k=1}^{j-1} \widehat{b_{i,k}}.\widehat{b_{j,k}} \right) \quad (1)$$

intervals during arithmetic operations (sum, subtraction, multiplication, etc.), which increases the number of operations required. If n is the size of the matrix, the complexity is necessarily greater than the order $O(n^3)$.
Operations with intervals can be more costly in terms of calculation time. Interval propagation during Cholesky decomposition may require additional computations to maintain the validity of the intervals. This can lead to an increase in execution time compared to the case of real matrices.

# 4 Application and comparison

## 4.1 Application

We take the symmetric matrix :
$\widehat{A} = \begin{pmatrix} [3.7;4.3] & [-1.5;-0.5] & [0;0] \\ [-1.5;-0.5] & [3.7;4.3] & [-1.5;-0.5] \\ [0;0] & [-1.5;-0.5] & [3.7;4.3] \end{pmatrix}$ and $\widehat{B} = \begin{pmatrix} [-14;0] \\ [-9;0] \\ [-3;0] \end{pmatrix}$ in example 1, we have shown that $\widehat{A}$ is symmetric positive definite, so we can decompose it as a product of a triangular matrix by its transpose using the decomposition of **Cholesky**

$$\widehat{F} = \begin{pmatrix} \widehat{b_{1,1}} & \widehat{0} & \widehat{0} \\ \widehat{b_{2,1}} & \widehat{b_{2,2}} & \widehat{0} \\ \widehat{b_{3,1}} & \widehat{b_{3,2}} & \widehat{b_{3,3}} \end{pmatrix} = \begin{pmatrix} [1.9;2.07] & [0;0] & [0;0] \\ [-0.76;-0.24] & [1.76;2.06] & [0;0] \\ [0;0] & [-0.8;-0.24] & [1.74;2.06] \end{pmatrix}$$

:
We can notice that:
$\widehat{F} \times \widehat{F}^T \approx \begin{pmatrix} [3.60;4.27] & [-1.52;-0.45] & [0;0] \\ [-1.52;-0.45] & [3.15;4.64] & [-1.56;-0.42] \\ [0;0] & [-1.56;-0.42] & [3.08;4.67] \end{pmatrix} \approx \widehat{A}$
let's put $\widehat{F}^T \widehat{X} = \widehat{Y}$ and we solve the system $\widehat{F}\widehat{Y} = \widehat{B}$, we find :

$$\widehat{Y} = \begin{pmatrix} [-7.05;0] \\ [-6.54;0] \\ [-3.35;0] \end{pmatrix}$$

System resolution $\widehat{A}\widehat{X} = \widehat{B}$ is to solve $\widehat{F}^T \widehat{X} = \widehat{Y}$.

$$\widehat{F}^T \widehat{X} = \widehat{Y} \text{ so } \widehat{X} = \begin{pmatrix} [-4.53;0] \\ [-3.9;0] \\ [-1.76;0] \end{pmatrix}$$

**Step 3** : We set $\widehat{F}^T \widehat{X} = \widehat{Y}$.
**Step 4** : Solve the system $\widehat{F}\widehat{Y} = \widehat{B}$.
**Step 5** : Solve the system $\widehat{F}^T \widehat{X} = \widehat{Y}$.

**Complexity of Cholesky decomposition**

The Cholesky decomposition for matrices with interval coefficients has specific characteristics compared with the case of real matrices. In terms of complexity, the Cholesky decomposition for matrices with interval coefficients is generally higher than for real matrices. This is due to the need to manipulate

**Benmohamed Khier** using Gauss elimination [6]
We notice that the method of **Choleski** with the arithmetic of the intervals gives more precise results and is very close compared to the others.

# 5 An application in input-output (I-O) model

## 5.1 The Leontief model

Leontief's model [1] helps analyze inter-industry production and economic relationships in an economy. The model assumes that each industry uses a combination of goods and services produced by other industries to produce its own goods and services.
Leontief's model uses an input-output matrix, also known as an "input-output matrix," which shows the quantity of each product needed to produce one unit of each final product. This matrix is used to calculate inter-industry linkages and multiplier effects in the economy. The Leontief model can be used to assess the impact of disruptions on specific industries or on the economy as a whole. It can also be used to assess the impact of economic policies, such as industrial development policies or trade policies.

## 5.2 A symmetric input-output matrix

A symmetric input-output matrix is one in which the quantity of each final product needed to produce a unit is the same regardless of the final product under consideration.
In a symmetric input-output matrix, the diagonal elements represent the share of total output that each industry uses to produce its own product. The off-diagonal elements represent the quantities of each product needed to produce one unit of each final product.
A symmetric input-output matrix represents an economy in which all industries are equally interrelated and interdependent. In other words, each industry depends on other industries to produce its own products, and each industry also contributes to the production of other industries, which can have a larger multiplier effect in the economy. Indeed, a disruption in one industry can have an impact on the entire economy, as each industry is closely linked to other industries.

### 4.2 Comparison of results:

$\widehat{X}=\begin{pmatrix}[-6.38;0.]\\ [-6.40;1.32]\\ [-3.40;0]\end{pmatrix}$ the result found by **Ning et al** using Gauss elimination [4]

$\widehat{X}=\begin{pmatrix}[-6.38;1.12]\\ [-6.40;1.54]\\ [-3.40;1.40]\end{pmatrix}$ the result found by **Ning et al** using **Hansen**'s technique [5]

$\widehat{X}=\begin{pmatrix}[-4.482;0]\\ [-3.816;0]\\ [-1.776;0.006]\end{pmatrix}$ the result found by **karkar nora -**

### 5.3 Input-output matrix with interval coefficient

If the coefficients of the input-output matrix are substituted with intervals instead of precise numerical values, it means that the exact quantities of each product needed to produce one unit of each final product are unknown or uncertain. The intervals may represent a range of possible values or uncertainty about the exact quantities needed. In this case, the interpretation of the input-output matrix must be modified accordingly. Instead of representing exact quantities, the input-output matrix represents qualitative relationships between industries and products. The interval coefficients can be used to perform sensitivity analyses or simulations to assess the impact of different uncertainties on the economy. For example, by modifying the intervals, it is possible to determine how changes in the output of one industry affect overall output and other industries. However, results obtained from an input-output matrix with interval coefficients may be less accurate than those obtained from a matrix with accurate numerical coefficients. It is therefore important to take into account the margin of error associated with the intervals when interpreting the results obtained from this matrix.

### 5.4 Application

Consider the input-output table (**table 1**) for this economic system with 3 industries A, B and C, where the coefficients are represented by intervals:
Let

**Table 1.** Input-output table

| ***** | A | B | C | Z |
|---|---|---|---|---|
| A | [0.1 ;0.2] | [0.2 ;0.3] | [0.1 ;0.2] | [7;9] |
| B | [0.2 ;0.3] | [0.3 ;0.4] | [0.2 ;0.3] | [2;3] |
| C | [0.1 ;0.2] | [0.2 ;0.3] | [0.3 ;0.4] | [0;0] |

- $\widehat{X}$ : be the production matrix
- $\widehat{A}$ : the domestic consumption matrix
- $\widehat{Z}$ : the export matrix
- $\widehat{I}$ : the identity matrix

To determine the level of production we solve the system

$$\widehat{X}=\widehat{AX}+\widehat{Z} \qquad (3)$$

$$(3)\Leftrightarrow \widehat{Z}=(\widehat{I}-\widehat{A})\widehat{X}$$

We put :

$$\widehat{B}=(\widehat{I}-\widehat{A})=\begin{pmatrix}[0.8;0.9] & [-0.3;-0.2] & [-0.2;-0.1]\\ [-0.3;-0.2] & [0.6;0.7] & [-0.3;-0.2]\\ [-0.2;-0.1] & [-0.3;-0.2] & [0.6;0.7]\end{pmatrix}:$$

Using the Sylvester criterion

$|[0.8;0.9]|=[0.8;0.9]>0$ ,

$$\begin{vmatrix}[0.8;0.9] & [-0.3;-0.2]\\ [-0.3;-0.2] & [0.6;0.7]\end{vmatrix}=[0.4;0.58]>0$$

And we have :

$$det(\widehat{B})=\begin{vmatrix}[0.8;0.9] & [-0.3;-0.2] & [-0.2;-0.1]\\ [-0.3;-0.2] & [0.6;0.7] & [-0.3;-0.2]\\ [-0.2;-0.1] & [-0.3;-0.2] & [0.6;0.7]\end{vmatrix}$$

$=[0.12;0.33]>0$

The matrix $\widehat{B}$ verifies the Sylvester criterion so it admits a Cholesky decomposition.

solve the system $\widehat{B}\widehat{X}=\widehat{Z}$

We find :

$$\widehat{X}=\begin{pmatrix}[9.71;19.98]\\ [6.61;19.33]\\ [3.15;13.85]\end{pmatrix}$$

# 6 Conclusions

The **Cholesky** decomposition is a numerically stable method to efficiently solve linear systems with interval coefficients. It also reduces the number of operations required to solve the system compared to other methods such as the Gauss-Jordan and Hansen methods.

In summary, the **Cholesky** decomposition is an important method for solving linear systems with interval coefficients because it guarantees the positive definition of the matrix, guarantees that the solution is also an interval, and can solve the system efficiently and numerically stable.

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Applying the algorithm, we get:
$\widehat{B} = \widehat{F} \times \widehat{F}^T$

With $\widehat{F}= \begin{pmatrix} [0.89;0.94] & [0;0] & [0;0] \\ [-0.33;-0.21] & [0.7;0.8] & [0;0] \\ [-0.22;-0.11] & [-0.5;-0.27] & [0.57;0.78] \end{pmatrix}$

To determine the level of production of each sector, we must

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# Convergence of the Jordan Neutrosophic Ideal in Neutrosophic Normed Spaces

**R. Muthuraj1, K. Nachammal1, M. Jeyaraman2,,D. Poovaragavan3**
1Department of Mathematics, H.H. The Rajah's College, Pudukkottai, Affiliated to Bharathidasan University, India
2Department of Mathematics, Raja Doraisingam Govt. Arts College, Sivagangai, Affiliated to Alagappa University, India
3Department of Mathematics, Government Arts College for Women, Sivagangai, Affiliated to Alagappa University, India

## A B S T R A C T

In the context of the Neutrosophic Norm, the essay explores the challenge of constructing precise sequence spaces whose elements' convergence is a generalised form of the Cauchy convergence. It has proven to be a crucial tool, opening the door to the theory of functions and the law of large numbers applications. Numerous authors, including those who investigated the Euler totient matrix operator, have studied the strategy for building new sequence spaces that are specified as the domain of matrix operators. Recently, the Jordan totient function Tr. generalised the Euler totient function ϕ. In the context of neutrosophic Norm spaces, we establish some sequence spaces, specifically cI 0(J,G,H)(Mr), cI (J,G,H)(Mr), ℓI ∞(J,G,H)(Mr) and ℓ∞(J,G,H)(Mr) as a domain of the triangular Jordan totient matrix operator, and investigate the ideal convergence of these sequences. These concepts serve as an introduction to a new sort of convergence that Fast and Steinhaus presented as more general than normal convergence and statistical convergence. According to Kostyrko et al., this form is known as ideal convergence. In order to arrive at a finite limit, the Jordan totient operator, an infinite matrix operator, is used. We also construct a number of inclusion connections between the spaces as we explain various topological and algebraic properties.
The Jordan totient operator, an infinite matrix operator, is used to accomplish the task of reaching a finite limit. As we discuss various topological and algebraic features, we also create several inclusion relations between the spaces.

**Keywords** Jordan I-convergence, Compactness, Completeness, Hausdorff, Neutrosophic Sets

## 1. Introduction

Fast [1] and Schoenberg [2] were the first to independently introduce the idea of statistical convergence. Salat et al. [3] established the concept of I-convergence, a statistical convergence generality. Later, the concept of statistical convergence for double sequences was independently developed by Edely and Mursaleen [4] and Tripathy[5], and for fuzzy numbers by Mursaleen and Saves [6]. In connection with this, there are two very distinct types of convergence for double sequences, namely I and I-convergence [7].

Converged triple sequences were introduced by Gurdal, Sahiner, and Duden [8] in 2007. Numerous authors have further explored this idea; see [9, 10, 11]. The I-convergence of triple sequences in probabilistic normed spaces was a concept used by Tripathy and Goswami[12].

A generalization of fuzzy set theory, intuitionistic fuzzy set theory was first proposed by Atanassov[13] in 1986. Fuzzy set theory is a powerful tool for modeling uncertainty and vagueness because it assigns the degree of membership to the components so that distinct individuals can be identified in a given set. The notion of fuzzy sets has curiously evolved into the present standard for young scientists or researchers, according to a large body of research that has lately arisen in the scientific discipline. The idea of fuzzy topology has become a highly important tool for many writers' works.

After some time, Smarandache [14], by introducing an inter mediate membership function, introduced the idea of Neutrosophic Sets [NS], which is a unique sort of notation for classical set theory. Thisset isaformal settingdesignedto gaugethe veracity ,ambiguity,and falsity of statements .Converged triple sequences were introduced byGurdal,Sahiner, andDudenin2007. Numerous authors have further explored this idea . The concept of triple sequences I-convergent in probabilistic normed spaces is familiar toTripathy and Goswami. Tripathy and Shiner examined the I-convergence qualities in triple sequence spaces and presented some in sightful findings.

The Jordan totient matrix operator, represented by Mr, is one such definite matrix operator. It was first described in [15] via the Jordan totient Tr function, whose domain and co domain are N.

$1 \leq \mathscr{f}_i \leq \mathfrak{z}$. It has the following definition: $\mathcal{T}_r(\mathfrak{z}) = \mathfrak{z}^r \prod_{p|\mathfrak{z}} \left(1 - \frac{1}{p^r}\right)$.
Since $\vartheta \geq 1$ is the prime decomposition of $\mathfrak{z}$, $\mathscr{f} = p_1^{\vartheta_1}, p_2^{\vartheta_2}, \cdots, p_{\hbar}^{\vartheta_{\hbar}}$. Consequently, the definition of the Jordan totient matrix operator $\mathcal{M}^r = (\rho^r_{\mathfrak{z}\hbar})$ is as follows:

$$\rho^r_{\mathfrak{z}\hbar} = \begin{cases} \dfrac{\mathcal{T}_r(\hbar)}{\mathfrak{z}^r} & \text{if } \hbar|\mathfrak{z}, \\ 0 & \text{otherwise,} \end{cases} \tag{1.1}$$

and its inverse $(\mathcal{M}^r)^{-1}$ is given by

$$(\mathcal{M}^r)^{-1} = \begin{cases} \dfrac{\mathcal{J}\left(\dfrac{\mathfrak{z}}{\hbar}\right)}{\mathcal{T}_r(\mathfrak{z})}\hbar^r & \text{if } \hbar|\mathfrak{z}, \\ 0 & \text{otherwise,} \end{cases} \tag{1.2}$$

where the Möbius function $\mathcal{J}$ is represented by the expression:

$$\mathcal{J}(\mathfrak{z}) = \begin{cases} 0 & \text{if } \; p^2|\mathfrak{z} \text{ for a few prime } p, \\ 1 & \text{if } \; \mathfrak{z} = 1, \\ (-1)^n & \text{if } \; \mathfrak{z} = \prod_{\hbar=1}^{n} p_{\hbar} \text{ where } p_{\hbar} s \text{ are vary.} \end{cases}$$

Later, using the operator to look at sequence spaces, Khan, Ilkhan, and Kara ([16], [17], and [18], respectively) presented some fascinating results. Also, we discussed Statistical $\Delta^{\mathfrak{m}}$-Convergence [19] and Lacunary $\Im$-Statistical Convergence [20] in Neutrosophic Normed Spaces. In this article, we construct sequence spaces, investigate the ideal convergence of these sequences, offer strong reasons against them, and analyze the algebraic and topological properties of these spaces within the framework of Neutrosophic Normed Spaces. Consider an open ball with a radius of $r > 0$, a center of $\psi$, and a fuzziness parameter of $\dot{\epsilon} \in (0, 1)$.

## 2 Preliminaries

For the two sequence spaces $\mathfrak{S}, \mathfrak{R}$ and an non finite matrix $\mathfrak{P} = (p_{\mathfrak{z}\mathfrak{k}})$, the $\mathfrak{P}$ transform of $\mathscr{D} = (\mathscr{D}_{\mathfrak{k}})$ provided by $\mathfrak{P}\mathscr{D} =$

*(e)* $\mathscr{J}(\alpha\dot{x}, \dot{\varpi}) = \mathscr{J}\left(\dot{x}, \frac{\dot{\varpi}}{|\alpha|}\right)$ *for every* $\alpha \neq 0$,

*(f)* $\mathscr{J}(\dot{x}, \dot{\varpi}) * \mathscr{J}(\dot{y}, \dot{\mathfrak{z}}) \leq \mathscr{J}(\dot{x} + \dot{y}, \dot{\varpi} + \dot{\mathfrak{z}})$,

*(g)* $\mathscr{J}(\dot{x}, \dot{\varpi}) : (0, \infty) \to [0, 1]$ *is continuous,*

*(h)* $\lim_{\dot{\varpi}\to\infty} \mathscr{J}(\dot{x}, \dot{\varpi}) = 1$ *and* $\lim_{\dot{\varpi}\to 0} \mathscr{J}(\dot{x}, \dot{\varpi}) = 0$,

*(i)* $\mathscr{G}(\dot{x}, \dot{\varpi}) < 1$,

*(j)* $\mathscr{G}(\dot{x}, \dot{\varpi}) = 0$ *if and only if* $\dot{x} = 0$,

*(k)* $\mathscr{G}(\alpha\dot{x}, \dot{\varpi}) = \mathscr{G}\left(\dot{x}, \frac{\dot{\varpi}}{|\alpha|}\right)$ *for each* $\alpha \neq 0$,

*(l)* $\mathscr{G}(\dot{x}, \dot{\varpi}) \odot \mathscr{G}(\dot{y}, \dot{\mathfrak{z}}) \geq \mathscr{G}(\dot{x} + \dot{y}, \dot{\varpi} + \dot{\mathfrak{z}})$,

*(m)* $\mathscr{G}(\dot{x}, \dot{\varpi}) : (0, \infty) \to [0, 1]$ *is continuous,*

*(n)* $\lim_{\dot{\varpi}\to\infty} \mathscr{G}(\dot{x}, \dot{\varpi}) = 0$ *and* $\lim_{\dot{\varpi}\to 0} \mathscr{G}(\dot{x}, \dot{\varpi}) = 1$,

*(o)* $\mathscr{H}(\dot{x}, \dot{\varpi}) < 1$,

*(p)* $\mathscr{H}(\dot{x}, \dot{\varpi}) = 0$ *if and only if* $\dot{x} = 0$,

*(q)* $\mathscr{H}(\alpha\dot{x}, \dot{\varpi}) = \mathscr{H}\left(\dot{x}, \frac{\dot{\varpi}}{|\alpha|}\right)$ *for each* $\alpha \neq 0$,

*(r)* $\mathscr{H}(\dot{x}, \dot{\varpi}) \Diamond \mathscr{G}(\dot{y}, \dot{\mathfrak{z}}) \geq \mathscr{H}(\dot{x} + \dot{y}, \dot{\varpi} + \dot{\mathfrak{z}})$

*(s)* $\mathscr{H}(\dot{x}, \dot{\varpi}) : (0, \infty) \to [0, 1]$ *is continuous,*

*(t)* $\lim_{\dot{\varpi}\to\infty} \mathscr{H}(\dot{x}, \dot{\varpi}) = 0$ *and* $\lim_{\dot{\varpi}\to 0} \mathscr{H}(\dot{x}, \dot{\varpi}) = 1$.

**Definition 2.4** *[20] Let* $(\Im, \mathscr{J}, \mathscr{G}, \mathscr{H}, *, \odot, \Diamond)$ *be an* $\mathcal{NNS}$. *A sequence* $\dot{x} = (\dot{x}_k)$ *is known as* $\mathcal{I}$*-convergent to* $\ell \in \Im$ *in regard to Neutrosophic Norms(*$\mathcal{NN}$*)* $(\mathscr{J}, \mathscr{G}, \mathscr{H})$*, if every* $\epsilon > 0$ *and* $\dot{\varpi} > 0$*, the set*

$$\left\{ \begin{array}{c} \mathscr{J}(\dot{x}_k - l, \dot{\varpi}) \leq 1 - \dot{\epsilon} \text{ or} \\ \mathscr{G}(\dot{x}_k - l, \dot{\varpi}) \geq \dot{\epsilon} \text{ and} \\ \mathscr{H}(\dot{x}_k - l, \dot{\varpi}) \geq \dot{\epsilon} \end{array} \right\} \in \mathcal{I}.$$

*(iii)* $\check{\mathfrak{U}} \in \mathfrak{F}$ *and* $\check{\mathfrak{V}} \supset \check{\mathfrak{U}}$ *implying* $\check{\mathfrak{V}} \in \mathfrak{F}$.

$\mathfrak{F}(\mathfrak{I}) = \{\mathscr{K} \subset \Im : \mathscr{K}^c \in \mathfrak{I}\}$ *is the filter connected to the ideal* $\mathfrak{I}$. *Consider* $\mathfrak{I}$ *as an admissible ideal in* $\mathbb{N}$.

**Definition 2.3** *[20] The 7-tuple* $(\Im, \mathscr{J}, \mathscr{G}, \mathscr{H}, *, \odot, \Diamond)$ *is known as* $\mathcal{NNS}$ *if* $\Im$ *is a linear space,* $*$ *is a continuous t-norm,* $\odot$ *and* $\Diamond$ *are continuous t-conorm,* $\mathscr{J}, \mathscr{G}$ *and* $\mathscr{H}$ *are fuzzy sets on* $\Im \times (0, \infty)$ *fulfils the coming after conditions:*
*For every one* $\dot{x}, \dot{y} \in \Im$ *and* $\dot{\mathfrak{z}}, \dot{\varpi} > 0$;

*(a)* $0 \leq \mathscr{J}(\dot{x}, \dot{\varpi}) \leq 1; 0 \leq \mathscr{G}(\dot{x}, \dot{\varpi}) \leq 1; 0 \leq \mathscr{H}(\dot{x}, \dot{\varpi}) \leq 1$,

*(b)* $\mathscr{J}(\dot{x}, \dot{\varpi}) + \mathscr{G}(\dot{x}, \dot{\varpi}) + \mathscr{H}(\dot{x}, \dot{\varpi}) \leq 3$,

*(c)* $\mathscr{J}(\dot{x}, \dot{\varpi}) > 0$,

*(d)* $\mathscr{J}(\dot{x}, \dot{\varpi}) = 1$ *if and only if* $\dot{x} = 0$,

## 3 Main results

For the purposes of this section, we'll assume that the ideal $I$ is a non-trivial admissible ideal of a subset of $\mathbb{N}$. Following sequence spaces are arranged as follows:

$$c^I_{(\mathscr{J},\mathscr{G},\mathscr{H})}(\mathcal{M}^r) = \left\{ \left\{ \begin{array}{c} n \in \mathbb{N} : \text{for a few } l \in \mathbb{C}, \\ \mathscr{J}(\mathcal{M}^r_n(p) - l, \dot{\varpi}) \leq 1 - \dot{\epsilon} \text{ or} \\ \mathscr{G}(\mathcal{M}^r_n(p) - l, \dot{\varpi}) \geq \dot{\epsilon} \text{ and} \\ \mathscr{H}(\mathcal{M}^r_n(p) - l, \dot{\varpi}) \geq \dot{\epsilon} \end{array} \right\} \in I \right\}, \quad (3.1)$$

$$c^I_{0(\mathscr{J},\mathscr{G},\mathscr{H})}(\mathcal{M}^r) = \left\{ \left\{ \begin{array}{c} \mathscr{J}(\mathcal{M}^r_n(p) - l, \dot{\varpi}) \leq 1 - \dot{\epsilon} \text{ or} \\ \mathscr{G}(\mathcal{M}^r_n(p) - l, \dot{\varpi}) \geq \dot{\epsilon} \text{ and} \\ \mathscr{H}(\mathcal{M}^r_n(p) - l, \dot{\varpi}) \geq \dot{\epsilon} \end{array} \right\} \in I \right\}, \quad (3.2)$$

$$\ell^I_{\infty(\mathscr{J},\mathscr{G},\mathscr{H})}(\mathcal{M}^r) = \left\{ \left\{ \begin{array}{c} \text{there exists } \dot{\epsilon} \in (0,1), \\ \mathscr{J}(\mathcal{M}^r_n(p), \dot{\varpi}) \leq 1 - \dot{\epsilon} \text{ or} \\ \mathscr{G}(\mathcal{M}^r_n(p), \dot{\varpi}) \geq \dot{\epsilon} \text{ and} \\ \mathscr{H}(\mathcal{M}^r_n(p), \dot{\varpi}) \geq \dot{\epsilon} \end{array} \right\} \in I \right\}, \quad (3.3)$$

$$\ell_{\infty(\mathscr{J},\mathscr{G},\mathscr{H})}(\mathcal{M}^r) = \left\{ \left\{ \begin{array}{c} n \in \mathbb{N} : \text{ there exists } \dot{\epsilon} \in (0,1), \\ \mathscr{J}(\mathcal{M}^r_n(p), \dot{\varpi}) \geq 1 - \dot{\epsilon} \text{ or} \\ \mathscr{G}(\mathcal{M}^r_n(p), \dot{\varpi}) \leq \dot{\epsilon} \text{ and} \\ \mathscr{H}(\mathcal{M}^r_n(p), \dot{\varpi}) \leq \dot{\epsilon} \end{array} \right\} \in I \right\}. \quad (3.4)$$

With regard to the fuzziness parameter $\dot{\epsilon} \in (0, 1)$, with a center at $p$ and a radius of $r > 0$, we offer the definitions of the open ball and closed ball as follows:

$$\mathfrak{B}^I_p(r, \epsilon)(\mathcal{M}^r) = \left\{ \begin{array}{c} \mathscr{J}(\mathcal{M}^r_n(p) - \mathcal{M}^r_n(\varphi), r) \leq 1 - \dot{\epsilon} \text{ or} \\ \mathscr{G}(\mathcal{M}^r_n(p) - \mathcal{M}^r_n(\varphi), r) \geq \dot{\epsilon} \text{ and} \\ \mathscr{H}(\mathcal{M}^r_n(p) - \mathcal{M}^r_n(\varphi), r) \geq \dot{\epsilon} \end{array} \right\} \in I \quad (3.5)$$

$\mathfrak{B}^I_p(r, \epsilon)(\mathcal{M}^r)$

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*The notation $\mathcal{I}_{(\mathscr{J},\mathscr{G},\mathscr{H})} - \lim \grave{x}_k = l$ will be used to indicate the ideal convergence of the sequence in the article $(\grave{x}_k)$ to $l$ in regard to the $\mathcal{NN}$ $(\mathscr{J},\mathscr{G},\mathscr{H})$.*

**Definition 2.5** *[20] Let $(\Im,\mathscr{J},\mathscr{G},\mathscr{H},*,\odot,\Diamond)$ be an $\mathcal{NNS}$. A sequence $\grave{x} = (\grave{x}_k)$ is known as $\mathcal{J}$-Cauchy sequence in regard to $(\mathscr{J},\mathscr{G},\mathscr{H})$, if every $\epsilon > 0$ and $\dot{\varpi} > 0$, there exists $p = p(\epsilon) \in \mathbb{N}$ like that the set*

$$\left\{ \begin{array}{c} \mathscr{J}(\grave{x}_k - \grave{x}_{\mathcal{N}}, \dot{\varpi}) \le 1 - \grave{\epsilon} \text{ or} \\ \mathscr{G}(\grave{x}_k - \grave{x}_{\mathcal{N}}, \dot{\varpi}) \ge \grave{\epsilon} \text{ and} \\ \mathscr{H}(\grave{x}_k - \grave{x}_{\mathcal{N}}, \dot{\varpi}) \ge \grave{\epsilon} \end{array} \right\} \in \mathcal{J}.$$

**Definition 2.6** *Let $(\Im,\mathscr{J},\mathscr{G},\mathscr{H},*,\odot,\Diamond)$ be an $\mathcal{NNS}$. Then $(\Im,\mathscr{J},\mathscr{G},\mathscr{H},*,\odot,\Diamond)$ is known as complete if each Cauchy sequence is convergent in regard to the $\mathcal{NN}$ $(\mathscr{J},\mathscr{G},\mathscr{H})$.*

sets:

$$\mathfrak{A} = \left\{ \begin{array}{c} \mathscr{J}\left(\mathcal{M}_n^r(\eta) - \eta_0, \frac{\dot{\varpi}}{2|\gamma|}\right) \le 1 - \grave{\epsilon} \text{ or} \\ \mathscr{G}\left(\mathcal{M}_n^r(\eta) - \eta_0, \frac{\dot{\varpi}}{2|\gamma|}\right) \ge \grave{\epsilon} \text{ and} \\ \mathscr{H}\left(\mathcal{M}_n^r(\eta) - \eta_0, \frac{\dot{\varpi}}{2|\gamma|}\right) \ge \grave{\epsilon} \end{array} \right\} \in I,$$

$$\mathfrak{A}^c = \left\{ \begin{array}{c} \mathscr{J}\left(\mathcal{M}_n^r(\eta) - \eta_0, \frac{\dot{\varpi}}{2|\gamma|}\right) > 1 - \grave{\epsilon} \text{ or} \\ \mathscr{G}\left(\mathcal{M}_n^r(\eta) - \eta_0, \frac{\dot{\varpi}}{2|\gamma|}\right) < \grave{\epsilon} \text{ and} \\ \mathscr{H}\left(\mathcal{M}_n^r(\eta) - \eta_0, \frac{\dot{\varpi}}{2|\gamma|}\right) < \grave{\epsilon} \end{array} \right\} \in \mathfrak{F}(I),$$

$$\mathfrak{B} = \left\{ \begin{array}{c} \mathscr{J}\left(\mathcal{M}_n^r(\chi) - \chi_0, \frac{\dot{\varpi}}{2|I|}\right) \le 1 - \grave{\epsilon} \text{ or} \\ \mathscr{G}\left(\mathcal{M}_n^r(\chi) - \chi_0, \frac{\dot{\varpi}}{2|I|}\right) \ge \grave{\epsilon} \text{ and} \\ \mathscr{H}\left(\mathcal{M}_n^r(\chi) - \chi_0, \frac{\dot{\varpi}}{2|I|}\right) \ge \grave{\epsilon} \end{array} \right\} \in I,$$

$$\mathfrak{B}^c = \left\{ \begin{array}{c} \mathscr{J}\left(\mathcal{M}_n^r(\chi) - \chi_0, \frac{\dot{\varpi}}{2|I|}\right) > 1 - \grave{\epsilon} \text{ or} \\ \mathscr{G}\left(\mathcal{M}_n^r(\chi) - \chi_0, \frac{\dot{\varpi}}{2|I|}\right) < \grave{\epsilon} \text{ and} \\ \mathscr{H}\left(\mathcal{M}_n^r(\chi) - \chi_0, \frac{\dot{\varpi}}{2|I|}\right) < \grave{\epsilon} \end{array} \right\} \in \mathfrak{F}(I),$$

The set $\mathscr{C} = \mathfrak{A}^c \cap \mathfrak{B}^c$ being a void set lies in $\mathfrak{F}(I)$, so consider $n \in \mathscr{C}$, then

$$\begin{aligned}
&\mathscr{J}\left(\mathcal{M}_n^r(\gamma\eta + I\chi) - (\gamma\eta_0 + I\chi_0), \dot{\varpi}\right) \\
&\ge \mathscr{J}\left(\gamma\mathcal{M}_n^r(\eta) - \gamma\eta_0, \frac{\dot{\varpi}}{2}\right) * \mathscr{J}\left(I\mathcal{M}_n^r(\chi) - I\chi_0, \frac{\dot{\varpi}}{2}\right) \\
&= \mathscr{J}\left(\mathcal{M}_n^r(\eta) - \eta_0, \frac{\dot{\varpi}}{2|\gamma|}\right) * \mathscr{J}\left(\mathcal{M}_n^r(\chi) - \chi_0, \frac{\dot{\varpi}}{2|I|}\right) \\
&> (1 - \grave{\epsilon}) * (1 - \grave{\epsilon}) = 1 - \grave{\epsilon} \\
&\Rightarrow \mathscr{J}\left(\mathcal{M}_n^r(\gamma\eta + I\chi) - (\gamma\eta_0 + I\chi_0), \dot{\varpi}\right) > 1 - \grave{\epsilon} \\
&\mathscr{G}\left(\mathcal{M}_n^r(\gamma\eta + I\chi) - (\gamma\eta_0 + I\chi_0), \dot{\varpi}\right) \\
&\le \mathscr{G}\left(\gamma\mathcal{M}_n^r(\eta) - \gamma\eta_0, \frac{\dot{\varpi}}{2}\right) \odot \mathscr{G}\left(I\mathcal{M}_n^r(\chi) - I\chi_0, \frac{\dot{\varpi}}{2}\right) \\
&= \mathscr{G}\left(\mathcal{M}_n^r(\eta) - \eta_0, \frac{\dot{\varpi}}{2|\gamma|}\right) \odot \mathscr{G}\left(\mathcal{M}_n^r(\chi) - \chi_0, \frac{\dot{\varpi}}{2|I|}\right) \\
&< \grave{\epsilon} \odot \grave{\epsilon} = \grave{\epsilon} \\
&\Rightarrow \mathscr{G}\left(\mathcal{M}_n^r(\gamma\eta + I\chi) - (\gamma\eta_0 + I\chi_0), \dot{\varpi}\right) < \grave{\epsilon} \\
&\mathscr{H}\left(\mathcal{M}_n^r(\gamma\eta + I\chi) - (\gamma\eta_0 + I\chi_0), \dot{\varpi}\right)
\end{aligned}$$

$$= \left\{ \begin{array}{c} \mathscr{J}(\mathcal{M}_n^r(p) - \mathcal{M}_n^r(\varphi), r) < 1 - \grave{\epsilon} \text{ or} \\ \mathscr{G}(\mathcal{M}_n^r(p) - \mathcal{M}_n^r(\varphi), r) > \grave{\epsilon} \text{ and} \\ \mathscr{H}(\mathcal{M}_n^r(p) - \mathcal{M}_n^r(\varphi), r) > \grave{\epsilon} \end{array} \right\} \in I \qquad (3.6)$$

**Theorem 3.1** *As of the Spaces $c^I_{0(\mathscr{J},\mathscr{G},\mathscr{H})}(\mathcal{M}^r)$ and $c^I_{(\mathscr{J},\mathscr{G},\mathscr{H})}(\mathcal{M}^r)$ are linear spaces.*

**Proof:**
The space $c^I_{0(\mathscr{J},\mathscr{G},\mathscr{H})}(\mathcal{M}^r)$'s linearity is demonstrated. On the basis of comparisons to $c^I_{(\mathscr{J},\mathscr{G},\mathscr{H})}(\mathcal{M}^r)$ may be drawn with certainty.
Given arbitrary sequences $\eta = (\eta_k), \chi = (\chi_k) \in c^I_{(\mathscr{J},\mathscr{G},\mathscr{H})}(\mathcal{M}^r)$ indicate that there is $\eta_0, \chi_0 \in \mathbb{C}$ so that $(\eta_k)$ and $(\chi_k)$ $I$-converge to $\eta_0$ and $\chi_0$, respectively.
For $\dot{\varpi} > 0, \grave{\epsilon} \in (0,1)$ and $\gamma, I \in \mathbb{R}$, think about the following

$\gamma\eta_0 + I\chi_0$.
Thus, $(\gamma\eta_k + I\chi_k) \in c^I_{(\mathscr{J},\mathscr{G},\mathscr{H})}(\mathcal{M}^r)$.
Therefore, the linear space is $c^I_{(\mathscr{J},\mathscr{G},\mathscr{H})}(\mathcal{M}^r)$.

**Theorem 3.2** *The inclusion relation $c^I_{0(\mathscr{J},\mathscr{G},\mathscr{H})}(\mathcal{M}^r) \subset c^I_{(\mathscr{J},\mathscr{G},\mathscr{H})}(\mathcal{M}^r) \subset c^I_{\infty(\mathscr{J},\mathscr{G},\mathscr{H})}(\mathcal{M}^r)$ holds.*

**Proof:**
The inclusion of $c^I_{0(\mathscr{J},\mathscr{G},\mathscr{H})}(\mathcal{M}^r)$ in $c^I_{(\mathscr{J},\mathscr{G},\mathscr{H})}(\mathcal{M}^r)$ is fairly obvious.
We provide evidence for $c^I_{(\mathscr{J},\mathscr{G},\mathscr{H})}(\mathcal{M}^r) \subset c^I_{\infty(\mathscr{J},\mathscr{G},\mathscr{H})}(\mathcal{M}^r)$.
Consider the sequence $p = (p_k) \in c^I_{(\mathscr{J},\mathscr{G},\mathscr{H})}(\mathcal{M}^r)$.
After it, there is $l \in \mathscr{C}$ like that
$I_{(\mathscr{J},\mathscr{G},\mathscr{H})}(\mathcal{M}^r) - \lim(p_k) = l$, and for every $\grave{\epsilon} \in (0,1)$ and $\dot{\varpi} > 0$, the set

$$\Delta = \left\{ \begin{array}{c} \mathscr{J}\left(\mathcal{M}_n^r(p) - l, \frac{\dot{\varpi}}{2}\right) > 1 - \grave{\epsilon} \text{ or} \\ \mathscr{G}\left(\mathcal{M}_n^r(p) - l, \frac{\dot{\varpi}}{2}\right) < \grave{\epsilon} \text{ and} \\ \mathscr{H}\left(\mathcal{M}_n^r(p) - l, \frac{\dot{\varpi}}{2}\right) < \grave{\epsilon} \end{array} \right\} \in F(I).$$

Choose $\mathscr{J}\left(l, \frac{\dot{\varpi}}{2}\right) = \mathfrak{p}, \mathscr{G}\left(l, \frac{\dot{\varpi}}{2}\right) = \mathfrak{q}$ and $\mathscr{H}\left(l, \frac{\dot{\varpi}}{2}\right) = \mathfrak{s}$ where $\mathfrak{p}, \mathfrak{q}, \mathfrak{s} \in (0,1), \dot{\varpi} > 0$ and $\grave{\epsilon} \in (0,1)$, there exist $c, d, e \in (0,1)$ like that $(1 - \grave{\epsilon}) * \mathfrak{p} > 1 - c, \grave{\epsilon} \odot \mathfrak{q} < d$ and $\grave{\epsilon} \Diamond \mathfrak{s} < e$.

For this reason $n \in \Im$, there are

$$\begin{aligned}
\mathscr{J}\left(\mathcal{M}_n^r(p), \dot{\varpi}\right) &= \mathscr{J}\left(\mathcal{M}_n^r(p) - l + l, \dot{\varpi}\right) \\
&\ge \mathscr{J}\left(\mathcal{M}_n^r(p) - l, \frac{\dot{\varpi}}{2}\right) * \mathscr{J}\left(l, \frac{\dot{\varpi}}{2}\right) \\
&> (1 - \grave{\epsilon}) * \mathfrak{p} > 1 - c, \\
\mathscr{G}\left(\mathcal{M}_n^r(p), \dot{\varpi}\right) &= \mathscr{G}\left(\mathcal{M}_n^r(p) - l + l, \dot{\varpi}\right) \\
&\le \mathscr{G}\left(\mathcal{M}_n^r(p) - l, \frac{\dot{\varpi}}{2}\right) \odot \mathscr{G}\left(l, \frac{\dot{\varpi}}{2}\right) \\
&< (1 - \grave{\epsilon}) \odot \mathfrak{q} < d, \\
\mathscr{H}\left(\mathcal{M}_n^r(p), \dot{\varpi}\right) &= \mathscr{H}\left(\mathcal{M}_n^r(p) - l + l, \dot{\varpi}\right) \\
&\le \mathscr{H}\left(\mathcal{M}_n^r(p) - l, \frac{\dot{\varpi}}{2}\right) \Diamond \mathscr{H}\left(l, \frac{\dot{\varpi}}{2}\right) \\
&< (1 - \grave{\epsilon}) \Diamond \mathfrak{s} < e.
\end{aligned}$$

Choose $\hbar = \max\{c, d, e\}$, there are

$$\leq \mathscr{H}\left(\gamma\mathcal{M}_n^r(\eta)-\gamma\eta_0,\frac{\dot{\varpi}}{2}\right)\Diamond\mathscr{H}\left(I\mathcal{M}_n^r(\chi)-I\chi_0,\frac{\dot{\varpi}}{2}\right)$$
$$=\mathscr{H}\left(\mathcal{M}_n^r(\eta)-\eta_0,\frac{\dot{\varpi}}{2|\gamma|}\right)\Diamond\mathscr{H}\left(\mathcal{M}_n^r(\chi)-\chi_0,\frac{\dot{\varpi}}{2|I|}\right)$$
$$<\grave{\epsilon}\Diamond\grave{\epsilon}=\grave{\epsilon}$$
$$\Rightarrow \mathscr{H}\left(\mathcal{M}_n^r(\gamma\eta+I\chi)-(\gamma\eta_0+I\chi_0),\dot{\varpi}\right)<\grave{\epsilon}$$

Then, we draw a conclusion
$$\mathscr{C}\subset\left\{\begin{array}{c}\mathcal{J}\left(\mathcal{M}_n^r(\gamma\eta+I\chi)-(\gamma\eta_0+I\chi_0),\dot{\varpi}\right)>1-\grave{\epsilon}\text{ or}\\ \mathcal{G}\left(\mathcal{M}_n^r(\gamma\eta+I\chi)-(\gamma\eta_0+I\chi_0),\dot{\varpi}\right)<\grave{\epsilon}\text{ and}\\ \mathscr{H}\left(\mathcal{M}_n^r(\gamma\eta+I\chi)-(\gamma\eta_0+I\chi_0),\dot{\varpi}\right)<\grave{\epsilon}\end{array}\right\}.$$
By utilising the attributes of $\mathfrak{F}(I)$, we have
$$\left\{\begin{array}{l}\mathcal{J}\left(\mathcal{M}_n^r(\gamma\eta+I\chi)-(\gamma\eta_0+I\chi_0),\dot{\varpi}\right)>1-\grave{\epsilon}\text{ or}\\ \mathcal{G}\left(\mathcal{M}_n^r(\gamma\eta+I\chi)-(\gamma\eta_0+I\chi_0),\dot{\varpi}\right)<\grave{\epsilon}\text{ and}\\ \mathscr{H}\left(\mathcal{M}_n^r(\gamma\eta+I\chi)-(\gamma\eta_0+I\chi_0),\dot{\varpi}\right)<\grave{\epsilon}\end{array}\right\}\in\mathfrak{F}(I),$$
which implies that the sequence $(\gamma\eta_k+I\chi_k)I$-converges to

**Example 3.4** *Assume that $(\mathbb{R},\|\cdot\|)$ is the normed space outfitted with the $\mathcal{NN}(\mathcal{J},\mathcal{G},\mathscr{H})$ as previously mentioned. Regarding the sequence $(p_k)=\sin\left(\frac{1}{k}\right)$.*
*Then, $(p_k)\in c^I_{\infty(\mathcal{J},\mathcal{G},\mathscr{H})}(\mathcal{M}^r)\setminus c^I_{(\mathcal{J},\mathcal{G},\mathscr{H})}(\mathcal{M}^r)$.*

**Theorem 3.5** *Each open ball has centre at $\psi$ and has positive radius $r$ in regard to parameter of fuzziness $\grave{\epsilon}$ lying between 0 and 1, i.e., $\mathfrak{B}^I_\psi(r,\grave{\epsilon})(\mathcal{M}^r)$ is an unclosed set in $c^I_{(\mathcal{J},\mathcal{G},\mathscr{H})}(\mathcal{M}^r)$.*

**Proof:**
Consider the open ball with $\psi$ as a center and a positive $r$ as a radius with the parameter of fuzziness $\grave{\epsilon}$ lies between 0 and 1,

$$\mathfrak{B}^I_\psi(r,\grave{\epsilon})(\mathcal{M}^r_n)$$
$$=\left\{\begin{array}{c}\mathcal{J}(\mathcal{M}^r_n(\psi)-\mathcal{M}^r_n(\Upsilon),r)\leq 1-\grave{\epsilon}\text{ or}\\ \mathcal{G}(\mathcal{M}^r_n(\psi)-\mathcal{M}^r_n(\Upsilon),r)\geq\grave{\epsilon}\text{ and}\\ \mathscr{H}(\mathcal{M}^r_n(\psi)-\mathcal{M}^r_n(\Upsilon),r)\geq\grave{\epsilon}\end{array}\right\}\in I$$
$$\mathfrak{B}^I_\psi(r,\grave{\epsilon})(\mathcal{M}^r_n)$$
$$=\left\{\begin{array}{c}\mathcal{J}(\mathcal{M}^r_n(\psi)-\mathcal{M}^r_n(\Upsilon),r)>1-\grave{\epsilon}\text{ or}\\ \mathcal{G}(\mathcal{M}^r_n(\psi)-\mathcal{M}^r_n(\Upsilon),r)<\grave{\epsilon}\text{ and}\\ \mathscr{H}(\mathcal{M}^r_n(\psi)-\mathcal{M}^r_n(\Upsilon),r)<\grave{\epsilon}\end{array}\right\}\in\mathfrak{F}(I).$$

Think about the element $\Upsilon=(\Upsilon_k)\in\mathfrak{B}^I_\psi(r,\grave{\epsilon})(\mathcal{M}^r_n)$.
Then its matching set

$$\left\{\begin{array}{c}\mathcal{J}(\mathcal{M}^r_n(\psi)-\mathcal{M}^r_n(\Upsilon),r)>1-\grave{\epsilon}\text{ or}\\ \mathcal{G}(\mathcal{M}^r_n(\psi)-\mathcal{M}^r_n(\Upsilon),r)<\grave{\epsilon}\text{ and}\\ \mathscr{H}(\mathcal{M}^r_n(\psi)-\mathcal{M}^r_n(\Upsilon),r)<\grave{\epsilon}\end{array}\right\}\in\mathfrak{F}(I).$$

For $\mathcal{J}(\mathcal{M}^r_n(\psi)-\mathcal{M}^r_n(\Upsilon),r)>1-\grave{\epsilon}$, $\mathcal{G}(\mathcal{M}^r_n(\psi)-\mathcal{M}^r_n(\Upsilon),r)<\grave{\epsilon}$ and $\mathscr{H}(\mathcal{M}^r_n(\psi)-\mathcal{M}^r_n(\Upsilon),r)<\grave{\epsilon}$ there exists $r_0$ lying between 0 and $r$ like that

$$\mathcal{J}(\mathcal{M}^r_n(\psi)-\mathcal{M}^r_n(\Upsilon),r_0)>1-\grave{\epsilon},$$
$$\mathcal{G}(\mathcal{M}^r_n(\psi)-\mathcal{M}^r_n(\Upsilon),r_0)<\grave{\epsilon}\text{ and}$$
$$\mathscr{H}(\mathcal{M}^r_n(\psi)-\mathcal{M}^r_n(\Upsilon),r_0)<\grave{\epsilon}.$$

$$\left\{\begin{array}{c}\mathcal{J}\left(\mathcal{M}^r_n(p)-l,\dot{\varpi}\right)>1-\hbar\text{ or}\\ \mathcal{G}\left(\mathcal{M}^r_n(p)-l,\dot{\varpi}\right)<\hbar\text{ and}\\ \mathscr{H}\left(\mathcal{M}^r_n(p)-l,\dot{\varpi}\right)<\hbar\end{array}\right\}\in\mathfrak{F}(I)$$
$$\Rightarrow p=(p_k)\in c^I_{\infty(\mathcal{J},\mathcal{G},\mathscr{H})}(\mathcal{M}^r).$$

The inclusion relation's inverse is not true. To prove our point, we offer the coming after examples.

**Example 3.3** *Let the Normed space $(\mathbb{R},\|\cdot\|)$ equipped with supremum norm, $r*p=\min\{r,p\}$, $r\odot p=\max\{r,p\}$ and $r\Diamond p=\max\{r,p\}$ for every $p,r\in(0,1)$. Consider the norms $(\mathcal{J},\mathcal{G},\mathscr{H})$ on $\Im^2\times(0,\infty)$ as follows: $\mathcal{J}(p,\dot{\varpi})=\frac{\dot{\varpi}}{\dot{\varpi}+\|p\|}$, $\mathcal{G}(p,\dot{\varpi})=\frac{\|p\|}{\dot{\varpi}+\|p\|}$, and $\mathscr{H}(p,\dot{\varpi})=\frac{\|p\|}{\dot{\varpi}}$. $(\mathbb{R},\mathcal{J},\mathcal{G},\mathscr{H},*,\odot,\Diamond)$ is then a standard $\mathcal{NNS}$. Regarding to the sequence $(p_k)=\left\{\frac{1}{k}+p_0\right\}$ where $p_0\in\mathbb{R}-\{0\}$. The sequence $(p_k)$ distinctly lying in $c^I_{(\mathcal{J},\mathcal{G},\mathscr{H})}(\mathcal{M}^r)\setminus c^I_{0(\mathcal{J},\mathcal{G},\mathscr{H})}(\mathcal{M}^r)$.*

Therefore,

$$\mathcal{J}(\mathcal{M}^r_n(\psi)-\mathcal{M}^r_n(\phi),r)$$
$$\geq\mathcal{J}(\mathcal{M}^r_n(\psi)-\mathcal{M}^r_n(\Upsilon),r_0)*\mathcal{J}(\mathcal{M}^r_n(\Upsilon)-\mathcal{M}^r_n(\phi),r-r_0)$$
$$\geq\grave{\epsilon}_0*\grave{\epsilon}_3\geq\grave{\epsilon}_0*\grave{\epsilon}_1>(1-s)>(1-\grave{\epsilon})$$
$$\Rightarrow\{n\in\mathbb{N}:\mathcal{J}(\mathcal{M}^r_n(\psi)-\mathcal{M}^r_n(\phi),r)>1-\grave{\epsilon}\}\in\mathfrak{F}(I),$$
$$\mathcal{G}(\mathcal{M}^r_n(\psi)-\mathcal{M}^r_n(\phi),r)$$
$$\leq\mathcal{G}(\mathcal{M}^r_n(\psi)-\mathcal{M}^r_n(\Upsilon),r_0)\odot\mathcal{G}(\mathcal{M}^r_n(\Upsilon)-\mathcal{M}^r_n(\phi),r-r_0)$$
$$\leq(1-\grave{\epsilon}_0)\odot(1-\grave{\epsilon}_4)\leq(1-\grave{\epsilon}_0)\odot(1-\grave{\epsilon}_2)<s<\grave{\epsilon}$$
$$\Rightarrow\{n\in\mathbb{N}:\mathcal{G}(\mathcal{M}^r_n(\psi)-\mathcal{M}^r_n(\phi),r)<\grave{\epsilon}\}\in\mathfrak{F}(I),$$

and correspondingly

$$\mathscr{H}(\mathcal{M}^r_n(\psi)-\mathcal{M}^r_n(\phi),r)$$
$$\leq\mathscr{H}(\mathcal{M}^r_n(\psi)-\mathcal{M}^r_n(\Upsilon),r_0)\Diamond\mathscr{H}(\mathcal{M}^r_n(\Upsilon)-\mathcal{M}^r_n(\phi),r-r_0)$$
$$\leq(1-\grave{\epsilon}_0)\Diamond(1-\grave{\epsilon}_4)\leq(1-\grave{\epsilon}_0)\Diamond(1-\grave{\epsilon}_2)<s<\grave{\epsilon}$$
$$\Rightarrow\{n\in\mathbb{N}:\mathscr{H}(\mathcal{M}^r_n(\psi)-\mathcal{M}^r_n(\phi),r)<\grave{\epsilon}\}\in\mathfrak{F}(I),$$

Hence, the set

$$\left\{\begin{array}{c}\mathcal{J}(\mathcal{M}^r_n(\psi)-\mathcal{M}^r_n(\phi),r)>1-\grave{\epsilon}\text{ or}\\ \mathcal{G}(\mathcal{M}^r_n(\psi)-\mathcal{M}^r_n(\phi),r)<\grave{\epsilon}\text{ and}\\ \mathscr{H}(\mathcal{M}^r_n(\psi)-\mathcal{M}^r_n(\phi),r)<\grave{\epsilon}\end{array}\right\}\in\mathfrak{F}(I)$$
$$\Rightarrow\mathfrak{B}^I_\Upsilon(r-r_0,1-\grave{\epsilon}_3)(\mathcal{M}^r_n)\subset\mathfrak{B}^I_\psi(r,\grave{\epsilon})(\mathcal{M}^r_n).$$

**Remark 3.6** *The spaces $c^I_{0(\mathcal{J},\mathcal{G},\mathscr{H})}(\mathcal{M}^r)$ and $c^I_{(\mathcal{J},\mathcal{G},\mathscr{H})}(\mathcal{M}^r)$ are $\mathcal{NNS}$ in regard to $\mathcal{NN}$ $(\mathcal{J},\mathcal{G},\mathscr{H})$.*

**Remark 3.7** *$\tau^I_{(\mathcal{J},\mathcal{G},\mathscr{H})}(\mathcal{M}^r)=\Big\{\mathfrak{A}\subset c^I_{(\mathcal{J},\mathcal{G},\mathscr{H})}(\mathcal{M}^r)$ : for each $\psi=(\psi_k)\in\mathfrak{A}$, a thing exists $r>0$ and $\grave{\epsilon}\in(0,1)$ like that $\mathfrak{B}^I_\psi(r,\grave{\epsilon})(\mathcal{M}^r_n)\subset\mathfrak{A}\Big\}$. Following that $\tau^I_{(\mathcal{J},\mathcal{G},\mathscr{H})}(\mathcal{M}^r)$ establishes a topology in the space of sequences $\tau^I_{(\mathcal{J},\mathcal{G},\mathscr{H})}(\mathcal{M}^r)$. The group is cited by*

$$\mathfrak{B}=\left\{\mathfrak{B}^I_\psi(r,\epsilon):\psi\in c^I_{(\mathcal{J},\mathcal{G},\mathscr{H})}(\mathcal{M}^r),r>0\,\text{and}\,\grave{\epsilon}\in(0,1)\right\}$$

*a foundation for the topology $\tau^I_{(\mathcal{J},\mathcal{G},\mathscr{H})}(\mathcal{M}^r)$ on the space*

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Setting $\grave{\epsilon}_0 = \mathcal{J}(\mathcal{M}_n^r(\psi) - \mathcal{M}_n^r(\Upsilon), r_0)$ we get $\grave{\epsilon}_0 > 1 - \grave{\epsilon}$ which further proves the element's existence $s \in (0,1)$ like that $\epsilon_0 > 1 - s > 1 - \grave{\epsilon}$.
For a certain $\grave{\epsilon}_0 > 1 - s$, we can locate $\grave{\epsilon}_1, \grave{\epsilon}_2, \grave{\epsilon}_3 \in (0,1)$ like that $\grave{\epsilon}_0 * \grave{\epsilon}_1 > 1 - s, (1 - \grave{\epsilon}_0) \odot (1 - \grave{\epsilon}_2) < s$ and $(1 - \grave{\epsilon}_0) \Diamond (1 - \grave{\epsilon}_3) < s$.
Assume $\grave{\epsilon}_4 = \max\{\grave{\epsilon}_1, \grave{\epsilon}_2, \grave{\epsilon}_3\}$.
Consider the open ball $\mathfrak{B}^I_{\Upsilon}(r - r_0, 1 - \grave{\epsilon}_4)(\mathcal{M}_n^r)$.
The restraint of $\mathfrak{B}^I_{\Upsilon}(r - r_0, 1 - \grave{\epsilon}_4)(\mathcal{M}_n^r)$ in $\mathfrak{B}^I_{\psi}(r, \grave{\epsilon})(\mathcal{M}_n^r)$ will provide the outcome we want.
Let $\phi = (\phi_k) \in \mathfrak{B}^I_{\Upsilon}(r - r_0, 1 - \grave{\epsilon}_4)(\mathcal{M}_n^r)$, then

$$\left\{ \begin{array}{c} \mathcal{J}(\mathcal{M}_n^r(\Upsilon) - \mathcal{M}_n^r(\phi), r - r_0) > \grave{\epsilon}_4 \text{ or} \\ \mathcal{G}(\mathcal{M}_n^r(\Upsilon) - \mathcal{M}_n^r(\phi), r - r_0) < 1 - \grave{\epsilon}_4 \text{ and} \\ \mathcal{H}(\mathcal{M}_n^r(\Upsilon) - \mathcal{M}_n^r(\phi), r - r_0) < 1 - \grave{\epsilon}_4 \end{array} \right\} \in \mathfrak{F}(I).$$

Then for every $\grave{\epsilon}_0 > \grave{\epsilon}$, there exist $\grave{\epsilon}_4, \grave{\epsilon}_5, \grave{\epsilon}_6 \in (0,1)$ like that $\grave{\epsilon}_4 * \grave{\epsilon}_4 \geq \grave{\epsilon}_0, (1 - \grave{\epsilon}_5) \odot (1 - \grave{\epsilon}_5) \leq (1 - \grave{\epsilon}_0)$ and $(1 - \grave{\epsilon}_6) \Diamond (1 - \grave{\epsilon}_6) \leq (1 - \grave{\epsilon}_0)$.
Assigning $\grave{\epsilon}_7 = \max\{\grave{\epsilon}_4, \grave{\epsilon}_5, \grave{\epsilon}_6\}$, consider the open balls $\mathfrak{B}^I_{\varphi}\left(1 - \grave{\epsilon}_7, \frac{r}{2}\right)(\mathcal{M}_n^r)$ and $\mathfrak{B}^I_{\phi}\left(1 - \grave{\epsilon}_7, \frac{r}{2}\right)(\mathcal{M}_n^r)$ centered at $\varphi$ and $\phi$ respectively.
We demonstrate that

$$\mathfrak{B}^I_{\varphi}\left(1 - \grave{\epsilon}_7, \frac{r}{2}\right)(\mathcal{M}_n^r) \cap \mathfrak{B}^I_{\phi}\left(1 - \grave{\epsilon}_7, \frac{r}{2}\right)(\mathcal{M}_n^r) = \mathcal{J}.$$

Ideally,
$\psi = (\psi_k) \in \mathfrak{B}^I_{\varphi}\left(1 - \grave{\epsilon}_7, \frac{r}{2}\right)(\mathcal{M}^r) \cap \mathfrak{B}^I_{\phi}\left(1 - \grave{\epsilon}_7, \frac{r}{2}\right)(\mathcal{M}_n^r)$.
For the set, next $\{n \in \mathbb{N}\} \in \mathfrak{F}(I)$, we have

$$\begin{aligned} \grave{\epsilon}_1 &= \mathcal{J}(\mathcal{M}_n^r(\varphi) - \mathcal{M}_n^r(\phi), r) \\ &\geq \mathcal{J}\left(\mathcal{M}_n^r(\varphi) - \mathcal{M}_n^r(\psi), \frac{r}{2}\right) * \mathcal{J}\left(\mathcal{M}_n^r(\psi) - \mathcal{M}_n^r(\phi), \frac{r}{2}\right) \quad (3.7) \\ &> \grave{\epsilon}_7 * \grave{\epsilon}_7 \geq \grave{\epsilon}_3 * \grave{\epsilon}_3 \geq \grave{\epsilon}_0 > \grave{\epsilon}_1 \end{aligned}$$

$$\begin{aligned} \grave{\epsilon}_2 &= \mathcal{G}(\mathcal{M}_n^r(\varphi) - \mathcal{M}_n^r(\phi), r) \\ &\leq \mathcal{G}\left(\mathcal{M}_n^r(\varphi) - \mathcal{M}_n^r(\psi), \frac{r}{2}\right) \odot \mathcal{G}\left(\mathcal{M}_n^r(\psi) - \mathcal{M}_n^r(\phi), \frac{r}{2}\right) \quad (3.8) \\ &< \grave{\epsilon}_7 \odot \grave{\epsilon}_7 \leq \grave{\epsilon}_4 \odot \grave{\epsilon}_4 \leq \grave{\epsilon}_0 > \grave{\epsilon}_2 \end{aligned}$$

$$\begin{aligned} \grave{\epsilon}_3 &= \mathcal{H}(\mathcal{M}_n^r(\varphi) - \mathcal{M}_n^r(\phi), r) \\ &\leq \mathcal{H}\left(\mathcal{M}_n^r(\varphi) - \mathcal{M}_n^r(\psi), \frac{r}{2}\right) \Diamond \mathcal{H}\left(\mathcal{M}_n^r(\psi) - \mathcal{M}_n^r(\phi), \frac{r}{2}\right) \quad (3.9) \\ &< \grave{\epsilon}_7 \Diamond \grave{\epsilon}_7 \leq \grave{\epsilon}_5 \Diamond \grave{\epsilon}_5 \leq \grave{\epsilon}_0 > \grave{\epsilon}_3 \end{aligned}$$

Equation (3.7) brings about contradiction.
Therefore,
$\mathfrak{B}^I_{\varphi}\left(1 - \grave{\epsilon}_7, \frac{r}{2}\right)(\mathcal{M}_n^r) \cap \mathfrak{B}^I_{\phi}\left(1 - \grave{\epsilon}_7, \frac{r}{2}\right)(\mathcal{M}_n^r) = \mathcal{J}$.
Hence, the space $c^I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r)$ is a Hausdorff space.

# 4 Jordan Neutrosophic Ideal Convergence

**Theorem 4.1** *If a sequence $\psi = (\psi_k) \in \xi$ is Jordan Neutrosophic Ideal Convergent ($\mathcal{JNIC}$) then the $I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}_n^r)$-limit* $c^I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r)$.

**Theorem 3.8** *The spaces $c^I_{0(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r)$ and $c^I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r)$ are Hausdorff spaces.*

**Proof:**
Let $\varphi = (\varphi_k)$ and $\phi = (\phi_k) \in c^I_{0(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r)$ such that $\varphi \neq \phi$.
Then every $n \in \mathbb{N}$ and $r > 0$, suggests

$$\begin{aligned} &0 < \mathcal{J}(\mathcal{M}_n^r(\varphi) - \mathcal{M}_n^r(\phi), r) < 1, \\ &0 < \mathcal{G}(\mathcal{M}_n^r(\varphi) - \mathcal{M}_n^r(\phi), r) < 1 \text{ and} \\ &0 < \mathcal{H}(\mathcal{M}_n^r(\varphi) - \mathcal{M}_n^r(\phi), r) < 1. \end{aligned}$$

Putting $\grave{\epsilon}_1 = \mathcal{J}(\mathcal{M}_n^r(\varphi) - \mathcal{M}_n^r(\phi), r), \grave{\epsilon}_2 = \mathcal{G}(\mathcal{M}_n^r(\varphi) - \mathcal{M}_n^r(\phi), r), \grave{\epsilon}_3 = \mathcal{H}(\mathcal{M}_n^r(\varphi) - \mathcal{M}_n^r(\phi), r)$ and $\grave{\epsilon} = \max\{\grave{\epsilon}_1, 1 - \grave{\epsilon}_2, 1 - \grave{\epsilon}_3\}$.

$$\mathfrak{R} = \left\{ \begin{array}{c} \mathcal{J}\left(\mathcal{M}_n^r(\psi) - l_2, \frac{\dot{\varpi}}{2}\right) \leq 1 - \grave{\epsilon}_1 \text{ or} \\ \mathcal{G}\left(\mathcal{M}_n^r(\psi) - l_2, \frac{\dot{\varpi}}{2}\right) \geq \grave{\epsilon}_1 \text{ and} \\ \mathcal{H}\left(\mathcal{M}_n^r(\psi) - l_2, \frac{\dot{\varpi}}{2}\right) \geq \grave{\epsilon}_1 \end{array} \right\} \in I$$

$$\mathfrak{R}^c = \left\{ \begin{array}{c} \mathcal{J}\left(\mathcal{M}_n^r(\psi) - l_2, \frac{\dot{\varpi}}{2}\right) > 1 - \grave{\epsilon}_1 \text{ or} \\ \mathcal{G}\left(\mathcal{M}_n^r(\psi) - l_2, \frac{\dot{\varpi}}{2}\right) < \grave{\epsilon}_1 \text{ and} \\ \mathcal{H}\left(\mathcal{M}_n^r(\psi) - l_2, \frac{\dot{\varpi}}{2}\right) < \grave{\epsilon}_1 \end{array} \right\} \in \mathfrak{F}(I)$$

Then $\mathfrak{S}^c \cap \mathfrak{R}^c \neq \mathcal{J}$. Taking $n \in \mathfrak{S}^c \cap \mathfrak{R}^c$, we have

$$\begin{aligned} &\mathcal{J}(l_1 - l_2, \dot{\varpi}) \\ &\geq \mathcal{J}\left(\mathcal{M}_n^r(\psi) - l_1, \frac{\dot{\varpi}}{2}\right) * \mathcal{J}\left(\mathcal{M}_n^r(\psi) - l_2, \frac{\dot{\varpi}}{2}\right) \\ &> (1 - \grave{\epsilon}_1) * (1 - \grave{\epsilon}_1) > (1 - \grave{\epsilon}), \\ &\mathcal{G}(l_1 - l_2, \dot{\varpi}) \\ &\leq \mathcal{G}\left(\mathcal{M}_n^r(\psi) - l_1, \frac{\dot{\varpi}}{2}\right) \odot \mathcal{G}\left(\mathcal{M}_n^r(\psi) - l_2, \frac{\dot{\varpi}}{2}\right) \\ &< \grave{\epsilon}_1 \odot \grave{\epsilon}_1 < \grave{\epsilon} \text{ and} \\ &\mathcal{H}(l_1 - l_2, \dot{\varpi}) \\ &\leq \mathcal{H}\left(\mathcal{M}_n^r(\psi) - l_1, \frac{\dot{\varpi}}{2}\right) \Diamond \mathcal{H}\left(\mathcal{M}_n^r(\psi) - l_2, \frac{\dot{\varpi}}{2}\right) \\ &< \grave{\epsilon}_1 \Diamond \grave{\epsilon}_1 < \grave{\epsilon} \end{aligned}$$

$\grave{\epsilon} \in (0,1)$ being arbitrary; $l_1 = l_2$. That is $I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}_n^r)$-limit is individual.

**Theorem 4.2** *A sequence of $\psi = (\psi_k) \in \xi$ is $\mathcal{JNIC}$ in regard to $\mathcal{NN}(\mathcal{J},\mathcal{G},\mathcal{H})$ if it is Jordan Neutrosophic Ideal Cauchy ($\mathcal{JNIC}a$) in relation to the same norms.*

**Proof:**
Let $\psi = (\psi_k) \in \xi$ be $\mathcal{JNIC}$ in regard to $\mathcal{NN}(\mathcal{J},\mathcal{G},\mathcal{H})$ like that
$I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}_n^r) - \lim(\psi_k) = l$ and there exists $\grave{\epsilon}_1 \in (0,1)$ like that
$(1 - \grave{\epsilon}_1) * (1 - \grave{\epsilon}_1) > 1 - \grave{\epsilon}, \grave{\epsilon}_1 \odot \grave{\epsilon}_1 < \grave{\epsilon}$ and $\grave{\epsilon}_1 \Diamond \grave{\epsilon}_1 < \grave{\epsilon}$ to a certain $\grave{\epsilon} \in (0,1)$.
Thus every $\dot{\varpi} > 0$,

$$\mathfrak{P} = \left\{ \begin{array}{c} \mathcal{J}(\mathcal{M}_n^r(\psi) - l_1, \dot{\varpi}) \leq 1 - \grave{\epsilon}_1 \text{ or} \\ \mathcal{G}(\mathcal{M}_n^r(\psi) - l_1, \dot{\varpi}) \geq \grave{\epsilon}_1 \text{ and} \\ \mathcal{H}(\mathcal{M}_n^r(\psi) - l_1, \dot{\varpi}) \geq \grave{\epsilon}_1 \end{array} \right\} \in I$$

*is unique.*

**Proof:**
Assuming that the $\mathcal{JNIC}$ sequence $\psi = (\psi_k)$ has non-identical ideal limits $l_1$ and $l_2$. There exists a $\grave{\epsilon}_1 \in (0,1)$ with the given $\grave{\epsilon} \in (0,1)$ like that
$(1-\grave{\epsilon}_1) * (1-\grave{\epsilon}_1) > 1-\grave{\epsilon}, \grave{\epsilon}_1 \odot \grave{\epsilon}_1 < \grave{\epsilon}$ and $\grave{\epsilon}_1 \Diamond \grave{\epsilon}_1 < \grave{\epsilon}$.
Hence, the sets

$$\mathfrak{S} = \left\{ \begin{array}{c} \mathcal{J}\left(\mathcal{M}_n^r(\psi) - l_1, \frac{\dot{\varpi}}{2}\right) \leq 1-\grave{\epsilon}_1 \text{ or} \\ \mathcal{G}\left(\mathcal{M}_n^r(\psi) - l_1, \frac{\dot{\varpi}}{2}\right) \geq \grave{\epsilon}_1 \text{ and} \\ \mathcal{H}\left(\mathcal{M}_n^r(\psi) - l_1, \frac{\dot{\varpi}}{2}\right) \geq \grave{\epsilon}_1 \end{array} \right\} \in I$$

$$\mathfrak{S}^c = \left\{ \begin{array}{c} \mathcal{J}\left(\mathcal{M}_n^r(\psi) - l_1, \frac{\dot{\varpi}}{2}\right) > 1-\grave{\epsilon}_1 \text{ or} \\ \mathcal{G}\left(\mathcal{M}_n^r(\psi) - l_1, \frac{\dot{\varpi}}{2}\right) < \grave{\epsilon}_1 \text{ and} \\ \mathcal{H}\left(\mathcal{M}_n^r(\psi) - l_1, \frac{\dot{\varpi}}{2}\right) < \grave{\epsilon}_1 \end{array} \right\} \in \mathfrak{F}(I)$$

$\mathcal{H}(\mathcal{M}_n^r(\psi) - \mathcal{M}_k^r(\psi), \dot{\varpi}) \geq \grave{\epsilon}$.
On the contrary, let $\mathcal{J}(\mathcal{M}_n^r(\psi) - \mathcal{M}_k^r(\psi), \dot{\varpi}) > 1-\grave{\epsilon}$. Then

$$\begin{aligned} 1-\grave{\epsilon} &\geq \mathcal{J}(\mathcal{M}_n^r(\psi) - \mathcal{M}_k^r(\psi), \dot{\varpi}) \\ &\geq \mathcal{J}\left(\mathcal{M}_n^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) * \mathcal{J}\left(\mathcal{M}_k^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) \\ &> (1-\grave{\epsilon}_1) * (1-\grave{\epsilon}_1) > (1-\grave{\epsilon}), \end{aligned}$$

this is incongruous.
Likewise, think about
$\mathcal{G}(\mathcal{M}_n^r(\psi) - \mathcal{M}_k^r(\psi), \dot{\varpi}) \geq \grave{\epsilon}, \mathcal{H}(\mathcal{M}_n^r(\psi) - \mathcal{M}_k^r(\psi), \dot{\varpi}) \geq \grave{\epsilon}$ such that
$\mathcal{G}\left(\mathcal{M}_n^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) \geq \grave{\epsilon}_1$ and $\mathcal{H}\left(\mathcal{M}_n^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) \geq \grave{\epsilon}_1$.
Contrarily, let $\mathcal{G}\left(\mathcal{M}_n^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) < \grave{\epsilon}_1$,
$\mathcal{H}\left(\mathcal{M}_n^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) < \grave{\epsilon}_1$. Hence

$$\begin{aligned} \epsilon &\leq \mathcal{G}(\mathcal{M}_n^r(\psi) - \mathcal{M}_k^r(\psi), \dot{\varpi}) \\ &\leq \mathcal{G}\left(\mathcal{M}_n^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) \odot \mathcal{G}\left(\mathcal{M}_k^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) \\ &< \grave{\epsilon}_1 \odot \grave{\epsilon}_1 < \grave{\epsilon}, \\ \epsilon &\leq \mathcal{H}(\mathcal{M}_n^r(\psi) - \mathcal{M}_k^r(\psi), \dot{\varpi}) \\ &\leq \mathcal{H}\left(\mathcal{M}_n^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) \Diamond \mathcal{H}\left(\mathcal{M}_k^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) \\ &< \grave{\epsilon}_1 \Diamond \grave{\epsilon}_1 < \grave{\epsilon}, \end{aligned}$$

it also contradicts itself.
Therefore, $n \in \mathbb{Q}$, we have
$\mathcal{J}(\mathcal{M}_n^r(\psi) - l, \dot{\varpi}) \leq 1-\grave{\epsilon}_1, \mathcal{G}(\mathcal{M}_n^r(\psi) - l, \dot{\varpi}) \geq \grave{\epsilon}_1$ and $\mathcal{H}(\mathcal{M}_n^r(\psi) - l, \dot{\varpi}) \geq \grave{\epsilon}_1$, may suggest $n \in \mathfrak{P}$.
Therefore, $\mathbb{Q} \subset \mathfrak{P}$ and $\mathbb{Q} \in I$.
As a result, the sequence $\psi = (\psi_k)$ is $\mathcal{JNIC}a$ in regard to norms $(\mathcal{J}, \mathcal{G}, \mathcal{H})$.
Conversely, let $\psi = (\psi_k)$ be $\mathcal{JNIC}a$ in regard to the norms $(\mathcal{J}, \mathcal{G}, \mathcal{H})$ and not $\mathcal{JNIC}$. Consequently, there is $k \in \mathbb{N}$ like that

$$\mathfrak{A} = \left\{ \begin{array}{c} \mathcal{J}(\mathcal{M}_n^r(\psi) - \mathcal{M}_k^r(\psi), \dot{\varpi}) \leq 1-\grave{\epsilon} \text{ or} \\ \mathcal{G}(\mathcal{M}_n^r(\psi) - \mathcal{M}_k^r(\psi), \dot{\varpi}) \geq \grave{\epsilon} \text{ and} \\ \mathcal{H}(\mathcal{M}_n^r(\psi) - \mathcal{M}_k^r(\psi), \dot{\varpi}) \geq \grave{\epsilon} \end{array} \right\} \in I$$

and

$$\mathfrak{P}^c = \left\{ \begin{array}{c} \mathcal{J}(\mathcal{M}_n^r(\psi) - l_1, \dot{\varpi}) > 1-\grave{\epsilon}_1 \text{ or} \\ \mathcal{G}(\mathcal{M}_n^r(\psi) - l_1, \dot{\varpi}) < \grave{\epsilon}_1 \text{ and} \\ \mathcal{H}(\mathcal{M}_n^r(\psi) - l_1, \dot{\varpi}) < \grave{\epsilon}_1 \end{array} \right\} \in \mathfrak{F}(I)$$

For $n \in \mathfrak{P}^c$,
$\mathcal{J}(\mathcal{M}_n^r(\psi) - l_1, \dot{\varpi}) > 1-\grave{\epsilon}_1$,
$\mathcal{G}(\mathcal{M}_n^r(\psi) - l_1, \dot{\varpi}) < \grave{\epsilon}_1$ and
$\mathcal{H}(\mathcal{M}_n^r(\psi) - l_1, \dot{\varpi}) < \grave{\epsilon}_1$.
For a certain $k \in \mathfrak{P}^c$, we may state

$$\mathbb{Q} = \left\{ \begin{array}{c} \mathcal{J}(\mathcal{M}_n^r(\psi) - \mathcal{M}_k^r(\psi), \dot{\varpi}) \leq 1-\grave{\epsilon} \text{ or} \\ \mathcal{G}(\mathcal{M}_n^r(\psi) - \mathcal{M}_k^r(\psi), \dot{\varpi}) \geq \grave{\epsilon} \text{ and} \\ \mathcal{H}(\mathcal{M}_n^r(\psi) - \mathcal{M}_k^r(\psi), \dot{\varpi}) \geq \grave{\epsilon} \end{array} \right\}.$$

Let $n \in \mathbb{Q} \Rightarrow \mathcal{J}(\mathcal{M}_n^r(\psi) - \mathcal{M}_k^r(\psi), \dot{\varpi}) \leq 1-\grave{\epsilon}$ or $\mathcal{G}(\mathcal{M}_n^r(\psi) - \mathcal{M}_k^r(\psi), \dot{\varpi}) \geq \grave{\epsilon}$ and

Simulteneously,

$$\begin{aligned} \grave{\epsilon} &\leq \mathcal{H}(\mathcal{M}_n^r(\psi) - \mathcal{M}_k^r(\psi), \dot{\varpi}) \\ &\leq \mathcal{H}\left(\mathcal{M}_n^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) \Diamond \mathcal{H}\left(\mathcal{M}_n^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) \\ &< \grave{\epsilon}_1 \Diamond \grave{\epsilon}_1 < \grave{\epsilon}. \end{aligned}$$

This resulted in conflict.
Therefore, $\mathfrak{B} \in \mathfrak{F}(I)$ and hence, $\psi = (\psi_k)$ is $\mathcal{JNIC}$.

**Theorem 4.3** *Consider $\mathcal{NNS}$ $c^I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r)$ and $\tau^I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r)$ be the topology on $c^I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r)$. Let $(\psi_j) = (\psi_k^j)_{j=1}^{\infty}$ be a sequence in $c^I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r)$. The sequence $\psi_j \to \psi$ as $j \to \infty$ if and only if $\mathcal{J}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), \dot{\varpi}) \to 1, \mathcal{G}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), \dot{\varpi}) \to 0$ and $\mathcal{H}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), \dot{\varpi}) \to 0$ as $n \to \infty$.*

**Proof:**
Let $\psi_j \to \psi$ as $j$ tends to $\infty$. Fix a specific $r > 0$ and $\grave{\epsilon} \in (0,1)$, there exists the natural number $n \in \mathbb{N}$ like that $(\psi_j) \in \mathfrak{B}^I_{\psi}(r, \grave{\epsilon})(\mathcal{M}_n^r)$ every $j \geq k$.
Then,

$$\mathfrak{S} = \left\{ \begin{array}{c} \mathcal{J}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), r) \leq 1-\grave{\epsilon} \text{ or} \\ \mathcal{G}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), r) \geq \grave{\epsilon} \text{ and} \\ \mathcal{H}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), r) \geq \grave{\epsilon} \end{array} \right\} \in I,$$

or equivalently,

$$\mathfrak{S}^c = \left\{ \begin{array}{c} \mathcal{J}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), r) > 1-\grave{\epsilon} \text{ or} \\ \mathcal{G}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), r) < \grave{\epsilon} \text{ and} \\ \mathcal{H}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), r) < \grave{\epsilon} \end{array} \right\} \in \mathfrak{F}(I)$$

For $\{n \in \mathbb{N}\} \subseteq \mathfrak{S}^c$,
$\mathcal{J}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), r) > 1-\grave{\epsilon}$
$\Rightarrow 1 - \mathcal{J}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), r) < \grave{\epsilon}$,
$\mathcal{G}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), r) < \grave{\epsilon}$ and
$\mathcal{H}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), r) < \grave{\epsilon}$.
Therefore, for $n \to \infty$,
$1 - \mathcal{J}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), r) \to 0$,
$\mathcal{G}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), r) \to 0$ and
$\mathcal{H}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), r) \to 0$.
This implies that

$$\mathfrak{B} = \left\{ \begin{array}{c} \mathcal{J}(\mathcal{M}_n^r(\psi) - l, \dot{\varpi}) > 1 - \grave{\epsilon} \text{ or} \\ \mathcal{G}(\mathcal{M}_n^r(\psi) - l, \dot{\varpi}) < \grave{\epsilon} \text{ and} \\ \mathcal{H}(\mathcal{M}_n^r(\psi) - l, \dot{\varpi}) < \grave{\epsilon} \end{array} \right\} \in I$$

$$\begin{aligned}
\Rightarrow 1 - \grave{\epsilon} &\geq \mathcal{J}(\mathcal{M}_n^r(\psi) - \mathcal{M}_k^r(\psi), \dot{\varpi}) \\
&\geq \mathcal{J}\left(\mathcal{M}_n^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) * \mathcal{J}\left(\mathcal{M}_n^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) \\
&> (1 - \grave{\epsilon}_1) * (1 - \grave{\epsilon}_1) > 1 - \grave{\epsilon} \text{ and} \\
\grave{\epsilon} &\leq \mathcal{G}(\mathcal{M}_n^r(\psi) - \mathcal{M}_k^r(\psi), \dot{\varpi}) \\
&\leq \mathcal{G}\left(\mathcal{M}_n^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) \odot \mathcal{G}\left(\mathcal{M}_n^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) \\
&< \grave{\epsilon}_1 \odot \grave{\epsilon}_1 < \grave{\epsilon},
\end{aligned}$$

Think about the ideal $I$ produced by the set $\{n \in \mathbb{N} : n < k\}$, it implies that the set's family of sets contains $\{n \in \mathbb{N} : n \geq k\}$ which relates to $\mathfrak{F}(I)$. Therefore,

$$\left\{ \begin{array}{c} \mathcal{J}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), \dot{\varpi}) > 1 - \grave{\epsilon} \text{ or} \\ \mathcal{G}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), \dot{\varpi}) < \grave{\epsilon} \text{ and} \\ \mathcal{H}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), \dot{\varpi}) < \grave{\epsilon} \end{array} \right\} \in \mathfrak{F}(I)$$

$$\Rightarrow (\psi_j) \in \mathfrak{B}_{\psi_k}^I(r, \grave{\epsilon})(\mathcal{M}_n^r), \text{ for every } n \geq k.$$

Thus, $\psi_j \to \psi$ as $j \to \infty$.

**Theorem 4.4** *Let $\psi = (\psi_k) \in c_{(\mathcal{J},\mathcal{G},\mathcal{H})}^I(\mathcal{M}^r)$. Then for anyone $l \in \mathcal{C}, \psi_k \xrightarrow{I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r)} l$ if for every $\grave{\epsilon} \in (0,1)$ and $\dot{\varpi} > 0$, positive integers exist $\mathcal{N} = \mathcal{N}(\psi, \grave{\epsilon}, \dot{\varpi})$ like that*

$$\left\{ \begin{array}{c} \mathcal{J}\left(\mathcal{M}_{\mathcal{N}}^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) > 1 - \grave{\epsilon} \text{ or} \\ \mathcal{G}\left(\mathcal{M}_{\mathcal{N}}^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) < \grave{\epsilon} \text{ and} \\ \mathcal{H}\left(\mathcal{M}_{\mathcal{N}}^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) < \grave{\epsilon} \end{array} \right\} \in \mathfrak{F}(I).$$

**Proof:**
Suppose $\psi_k \xrightarrow{I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r)} l$ for anyone $l \in \mathcal{C}$.
For given that $\grave{\epsilon} \in (0,1)$, there exists a decimal $r \in (0,1)$ like that $(1-\grave{\epsilon}) * (1-\grave{\epsilon}) > 1 - r, \grave{\epsilon} \odot \grave{\epsilon} < r$ with $\grave{\epsilon} \Diamond \grave{\epsilon} < r$.
Since $\psi_k \xrightarrow{I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{M}^r)} l$, for every $\dot{\varpi} > 0$,

$$\Delta = \left\{ \begin{array}{c} \mathcal{J}\left(\mathcal{M}_n^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) \leq 1 - \grave{\epsilon} \text{ or} \\ \mathcal{G}\left(\mathcal{M}_n^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) \geq \grave{\epsilon} \text{ and} \\ \mathcal{H}\left(\mathcal{M}_n^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) \geq \grave{\epsilon} \end{array} \right\} \in I;$$

which implies that

$$\Delta^c = \left\{ \begin{array}{c} \mathcal{J}\left(\mathcal{M}_n^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) > 1 - \grave{\epsilon} \text{ or} \\ \mathcal{G}\left(\mathcal{M}_n^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) < \grave{\epsilon} \text{ and} \\ \mathcal{H}\left(\mathcal{M}_n^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) < \grave{\epsilon} \end{array} \right\} \in \mathfrak{F}(I).$$

On the other hand, let us pick $\mathcal{N} \in \Delta^c$.
Then

$$\left\{ \begin{array}{c} \mathcal{J}\left(\mathcal{M}_{\mathcal{N}}^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) > 1 - \grave{\epsilon} \text{ or} \\ \mathcal{G}\left(\mathcal{M}_{\mathcal{N}}^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) < \grave{\epsilon} \text{ and} \\ \mathcal{H}\left(\mathcal{M}_{\mathcal{N}}^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) < \grave{\epsilon} \end{array} \right\}.$$

We demonstrate that a non negative integer exists $\mathcal{N} = \mathcal{N}(\psi, \epsilon, \dot{\varpi})$ like that
$\mathcal{J}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), r) \to 1$,
$\mathcal{G}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), r) \to 0$ and
$\mathcal{H}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), r) \to 0$ as $n \to \infty$.
On the other hand, imagine for every $\dot{\varpi} > 0$,
$\mathcal{J}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), \dot{\varpi}) \to 1$,
$\mathcal{G}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), \dot{\varpi}) \to 0$ and
$\mathcal{H}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), \dot{\varpi}) \to 0$ as $n$ tends to $\infty$.
Then every $\grave{\epsilon} \in (0,1)$, a thing exists the natural number $k \in \mathbb{N}$ like that
$1 - \mathcal{J}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), \dot{\varpi}) < \grave{\epsilon}$,
$\mathcal{G}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), \dot{\varpi}) < \grave{\epsilon}$ and
$\mathcal{H}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), \dot{\varpi}) < \grave{\epsilon}$ for every $n \geq k$
$\Rightarrow \mathcal{J}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), \dot{\varpi}) > 1 - \grave{\epsilon}$,
$\mathcal{G}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), \dot{\varpi}) < \grave{\epsilon}$ and
$\mathcal{H}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), \dot{\varpi}) < \grave{\epsilon}$ for each $n \geq k$.

This is incongruous. Also,

$$\mathcal{G}(\mathcal{M}_n^r(\psi) - \mathcal{M}_{\mathcal{N}}^r(\psi), \dot{\varpi}) \geq r \text{ and } \mathcal{G}(\mathcal{M}_n^r(\psi) - l, \dot{\varpi}) < \grave{\epsilon}.$$

Particularly, $\mathcal{G}(\mathcal{M}_n^r(\psi) - l, \dot{\varpi}) < \grave{\epsilon}$.
Therefore,

$$\begin{aligned}
r &\leq \mathcal{G}(\mathcal{M}_n^r(\psi) - \mathcal{M}_{\mathcal{N}}^r(\psi), \dot{\varpi}) \\
&\leq \mathcal{G}\left(\mathcal{M}_n^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) \odot \mathcal{G}\left(\mathcal{M}_{\mathcal{N}}^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) \\
&\leq \grave{\epsilon} \odot \grave{\epsilon} < r,
\end{aligned}$$

which again is in conflict. Similarly,

$$\mathcal{H}(\mathcal{M}_n^r(\psi) - \mathcal{M}_{\mathcal{N}}^r(\psi), \dot{\varpi}) \geq r \text{ and } \mathcal{H}(\mathcal{M}_n^r(\psi) - l, \dot{\varpi}) < \grave{\epsilon}.$$

Particularly, $\mathcal{H}(\mathcal{M}_n^r(\psi) - l, \dot{\varpi}) < \grave{\epsilon}$.
Therefore,

$$\begin{aligned}
r &\leq \mathcal{H}(\mathcal{M}_n^r(\psi) - \mathcal{M}_{\mathcal{N}}^r(\psi), \dot{\varpi}) \\
&\leq \mathcal{H}\left(\mathcal{M}_n^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) \Diamond \mathcal{H}\left(\mathcal{M}_{\mathcal{N}}^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) \\
&\leq \grave{\epsilon} \Diamond \grave{\epsilon} < r,
\end{aligned}$$

Therefore, $\mathfrak{P} \subseteq \Delta$ and since $\Delta \in I \Longrightarrow \mathfrak{P} \in I$.

**Definition 4.5** *The void set $\mathfrak{S} \subset c_{(\mathcal{J},\mathcal{G},\mathcal{H})}^I(\mathcal{M}^r)$ is compact if each open cover of $\mathfrak{S}$ specified by the open set of $\tau_{(\mathcal{J},\mathcal{G},\mathcal{H})}^I(\mathcal{M}^r)$ has a non infinite subcover.*

**Theorem 4.6** *Each non infinite subset $\mathfrak{S}$ of $c_{(\mathcal{J},\mathcal{G},\mathcal{H})}^I(\mathcal{M}^r)$ is compact.*

**Proof:**
Let $\mathfrak{S} = \{\psi_1, \psi_2, \psi_3, \ldots, \psi_n\}$ be the finite subset of $c_{(\mathcal{J},\mathcal{G},\mathcal{H})}^I(\mathcal{M}^r)$. For $r > 0$ and $0 < \grave{\epsilon} < 1$.
Let $\left\{\mathfrak{B}_\psi^I(r, \grave{\epsilon})(\mathcal{M}_n^r) : \psi \in \mathfrak{S}\right\}$ be an open cover of $\mathfrak{S}$.
Following that $\mathfrak{S} \subseteq \bigcup_{\psi \in \mathfrak{S}} \mathfrak{B}_\psi^I(r, \grave{\epsilon})(\mathcal{M}_n^r)$.
Now for each $\psi_i \in \mathfrak{S}, i = 1, 2, 3, \ldots n$, we have
$\psi_i \in \bigcup_{\psi_i \in s} \mathfrak{B}_{\psi_i}^I(r, \grave{\epsilon})(\mathcal{M}_n^r)$.

$$\mathfrak{P} = \left\{ \begin{array}{c} \mathcal{J}(\mathcal{M}_n^r(\psi) - \mathcal{M}_{\mathcal{N}}^r(\psi), \dot{\varpi}) \leq 1 - r \text{ or} \\ \mathcal{G}(\mathcal{M}_n^r(\psi) - \mathcal{M}_{\mathcal{N}}^r(\psi), \dot{\varpi}) \geq r \\ \mathcal{H}(\mathcal{M}_n^r(\psi) - \mathcal{M}_{\mathcal{N}}^r(\psi), \dot{\varpi}) \geq r \end{array} \right\} \in I.$$

We'll demonstrate that $\mathfrak{P} \subseteq \Delta$. Contrarily, let $\mathfrak{P} \not\subseteq \Delta$, that is, there exists $n \in \mathfrak{P}$ like that $n$ not in $\Delta$.
Then $\mathcal{J}(\mathcal{M}_n^r(\psi) - \mathcal{M}_{\mathcal{N}}^r(\psi), \dot{\varpi}) \leq 1 - r$ and $\mathcal{J}\left(\mathcal{M}_n^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) > 1 - \dot{\epsilon}$.
Particularly, $\mathcal{J}\left(\mathcal{M}_n^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) > 1 - \dot{\epsilon}$.
Therefore, we have

$$\begin{aligned} 1 - r &\geq \mathcal{J}(\mathcal{M}_n^r(\psi) - \mathcal{M}_{\mathcal{N}}^r(\psi), \dot{\varpi}) \\ &\geq \mathcal{J}\left(\mathcal{M}_n^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) * \mathcal{J}\left(\mathcal{M}_{\mathcal{N}}^r(\psi) - l, \frac{\dot{\varpi}}{2}\right) \\ &\geq (1 - \dot{\epsilon}) * (1 - \dot{\epsilon}) > 1 - r \end{aligned}$$

$(\psi_j) \in \mathfrak{B}_\psi^I\left(\frac{r}{3}, \dot{\epsilon}\right)(\mathcal{M}_n^r)$.
Therefore, the set

$$\Delta = \left\{ \begin{array}{c} \mathcal{J}\left(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), \frac{r}{3}\right) > 1 - \dot{\epsilon} \text{ or} \\ \mathcal{G}\left(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), \frac{r}{3}\right) < \dot{\epsilon} \\ \mathcal{H}\left(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), \frac{r}{3}\right) < \dot{\epsilon} \end{array} \right\} \in \mathfrak{F}(I).$$

A finite subcover exists $\left\{\mathfrak{B}_{\psi_i}^I\left(\frac{r}{3}, \dot{\epsilon}\right)(\mathcal{M}_n^r) : \psi_i \in \mathfrak{S} \text{ and } i = 1, 2, 3, \ldots, m\right\}$ since $\mathfrak{S}$ is compact like that $\mathfrak{S} \subseteq \bigcup_{i=1}^m \mathfrak{B}_{\psi_i}^I\left(\frac{r}{3}, \dot{\epsilon}\right)(\mathcal{M}_n^r)$.
Let $(\psi^{j_p})$ be a subsequence of $(\psi_j)$.
Then $(\psi^{j_p}) \in \bigcup_{i=1}^m \mathfrak{B}_{\psi_i}^I\left(\frac{r}{3}, \dot{\epsilon}\right)(\mathcal{M}_n^r)$, implies $(\psi^{j_p}) \in \mathfrak{B}_{\psi_i}^I\left(\frac{r}{3}, \dot{\epsilon}\right)(\mathcal{M}_n^r)$, for some $\psi_i \in \mathfrak{S}$.
Therefore, the set

$$\nabla = \left\{ \begin{array}{c} \mathcal{J}\left(\mathcal{M}_n^r(\psi^{j_p}) - \mathcal{M}_n^r(\psi_i), \frac{r}{3}\right) > 1 - \dot{\epsilon} \text{ or} \\ \mathcal{G}\left(\mathcal{M}_n^r(\psi^{j_p}) - \mathcal{M}_n^r(\psi_i), \frac{r}{3}\right) < \dot{\epsilon} \text{ and} \\ \mathcal{H}\left(\mathcal{M}_n^r(\psi^{j_p}) - \mathcal{M}_n^r(\psi_i), \frac{r}{3}\right) < \dot{\epsilon} \end{array} \right\} \in \mathfrak{F}(I).$$

For $n \in \Delta \cap \nabla$,

$$\begin{aligned} &\mathcal{J}(\mathcal{M}_n^r(\psi^{j_p}) - \mathcal{M}_n^r(\psi), r) \\ &\geq \mathcal{J}\left(\mathcal{M}_n^r(\psi^{j_p}) - \mathcal{M}_n^r(\psi_i), \frac{r}{3}\right) * \mathcal{J}\left(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi_i), \frac{r}{3}\right) \\ &* \mathcal{J}\left(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), \frac{r}{3}\right) \\ &> (1 - \dot{\epsilon}) * (1 - \dot{\epsilon}) * (1 - \dot{\epsilon}) = (1 - \dot{\epsilon}). \end{aligned}$$

Also

$$\begin{aligned} &\mathcal{G}(\mathcal{M}_n^r(\psi^{j_p}) - \mathcal{M}_n^r(\psi), r) \\ &\leq \mathcal{G}\left(\mathcal{M}_n^r(\psi^{j_p}) - \mathcal{M}_n^r(\psi_i), \frac{r}{3}\right) \odot \mathcal{G}\left(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi_i), \frac{r}{3}\right) \\ &\odot \mathcal{G}\left(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), \frac{r}{3}\right) \\ &< \dot{\epsilon} \odot \dot{\epsilon} \odot \dot{\epsilon} = \dot{\epsilon}. \end{aligned}$$

Simultaneously,

$$\begin{aligned} &\mathcal{H}(\mathcal{M}_n^r(\psi^{j_p}) - \mathcal{M}_n^r(\psi), r) \\ &\leq \mathcal{H}\left(\mathcal{M}_n^r(\psi^{j_p}) - \mathcal{M}_n^r(\psi_i), \frac{r}{3}\right) \lozenge \mathcal{H}\left(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi_i), \frac{r}{3}\right) \end{aligned}$$

That suggests $\psi_i \in \mathfrak{B}_{\psi_j}^I(r, \dot{\epsilon})(\mathcal{M}_n^r)$ for anyone $j \in \{1, 2, 3, \ldots, n\}$.
Following that $\left\{\mathfrak{B}_{\psi_i}^I(r, \dot{\epsilon})(\mathcal{M}_n^r) : i = 1, 2, 3, \ldots, n\right\}$ is a non infinite subcover of $\mathfrak{S}$.

**Theorem 4.7** *The set $\mathfrak{S} \subseteq c_{(\mathcal{J},\mathcal{G},\mathcal{H})}^I(\mathcal{M}^r)$ is compact if each sequence in $\mathfrak{S}$ has a convergent subsequence.*

**Proof:**
A compact subset of $c_{(\mathcal{J},\mathcal{G},\mathcal{H})}^I(\mathcal{M}^r)$ would be $\mathfrak{S}$, assuming that let $(\psi_k^j) = (\psi_j)_{j=1}^\infty$ be a sequence in $\mathfrak{S}$.
Given $0 < \dot{\epsilon} < 1$ and $r > 0$,
let $\left\{\mathfrak{B}_\psi^I\left(\frac{r}{3}, \dot{\epsilon}\right)(\mathcal{M}_n^r) : \psi = (\psi_k) \in \mathfrak{S}\right\}$ be an open cover of $\mathfrak{S}$.
This suggests, $(\psi_j) \in \bigcup_{\psi \in \mathfrak{S}} \left\{\mathfrak{B}_\psi^I\left(\frac{r}{3}, \dot{\epsilon}\right)(\mathcal{M}_n^r)\right\}$.
Next, there are some $\psi = (\psi_k) \in \mathfrak{S}$ such that hand, let $\left\{\mathfrak{B}_\psi^I\left(\frac{r}{3}, \dot{\epsilon}\right)(\mathcal{M}_n^r)\right\}$ be an open cover of $\mathfrak{S} \Longrightarrow \mathfrak{S} \subseteq \bigcup_{\psi \in \mathfrak{S}} \mathfrak{B}_\psi^I\left(\frac{r}{3}, \dot{\epsilon}\right)(\mathcal{M}_n^r)$.
Therefore, the set

$$\left\{ \begin{array}{c} \mathcal{J}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), r) > 1 - \dot{\epsilon} \text{ or} \\ \mathcal{G}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), r) < \dot{\epsilon} \text{ and} \\ \mathcal{H}(\mathcal{M}_n^r(\psi_j) - \mathcal{M}_n^r(\psi), r) < \dot{\epsilon} \end{array} \right\} \in \mathfrak{F}(I).$$

Due to $\mathfrak{S}$ not being compact, there exists a non infinite subcover $\left\{\mathfrak{B}_{\psi_i}^I(r, \dot{\epsilon})(\mathcal{M}_n^r) : \psi_i \in \mathfrak{S}, i = 1, 2, 3 \ldots, m\right\}$ like that $\mathfrak{S} \not\subseteq \bigcup_{\psi_i \in \mathfrak{S}} \mathfrak{B}_{\psi_i}^I(r, \dot{\epsilon})(\mathcal{M}_n^r)$, it suggests the set

$$\left\{ \begin{array}{c} \mathcal{J}(\mathcal{M}_n^r(\psi^{j_p}) - \mathcal{M}_n^r(\psi_i), r) > 1 - \dot{\epsilon} \text{ or} \\ \mathcal{G}(\mathcal{M}_n^r(\psi^{j_p}) - \mathcal{M}_n^r(\psi_i), r) < \dot{\epsilon} \text{ and} \\ \mathcal{H}(\mathcal{M}_n^r(\psi^{j_p}) - \mathcal{M}_n^r(\psi_i), r) < \dot{\epsilon} \end{array} \right\} \notin \mathfrak{F}(I)$$

$\Rightarrow$ for anyone $\dot{\epsilon} \in (0, 1)$ and a positive $r$, $(\psi^{j_p}) \notin \mathfrak{B}_\psi^I(r, \dot{\epsilon})$.
Hence, $(\psi^{j_p}) \nrightarrow z$. This is incongruous.
Thus, $\mathfrak{S}$ is compact.

**Theorem 4.8** *Consider the $\mathcal{NNS}$ $c_{(\mathcal{J},\mathcal{G},\mathcal{H})}^I(\mathcal{M}^r)$. Choose a positive $r$ and $0 < \dot{\epsilon}, \dot{\epsilon}' < 1$ like that $(1 - \epsilon') \leq (1 - \dot{\epsilon}) * (1 - \dot{\epsilon})$, $\dot{\epsilon} \odot \dot{\epsilon} \leq \dot{\epsilon}'$ and $\dot{\epsilon} \lozenge \dot{\epsilon} \leq \dot{\epsilon}'$. Then for some one $\psi = (\psi_k) \in c_{(\mathcal{J},\mathcal{G},\mathcal{H})}^I(\mathcal{M}^r)$, $\overline{\mathfrak{B}_\psi^I\left(\frac{r}{2}, \dot{\epsilon}\right)(\mathcal{M}_n^r)} \subseteq \mathfrak{B}_\psi^I(r, \dot{\epsilon}')(\mathcal{M}_n^r)$.*

**Proof:**
Let $q = (q_k) \in \mathfrak{B}_\psi^I\left(\frac{r}{2}, \dot{\epsilon}\right)(\mathcal{M}_n^r)$ and $\mathfrak{B}_q^I\left(\frac{r}{2}, \dot{\epsilon}\right)(\mathcal{M}_n^r)$ be an open ball which has centre at $q$ and has radius $\dot{\epsilon}$.
Thus, $\mathfrak{B}_\psi^I\left(\frac{r}{2}, \dot{\epsilon}\right)(\mathcal{M}_n^r) \cap \mathfrak{B}_q^I\left(\frac{r}{2}, \dot{\epsilon}\right)(\mathcal{M}_n^r) \neq \mathcal{J}$.
Suppose $\varphi = (\varphi_k) \in \mathfrak{B}_q^I\left(\frac{r}{2}, \dot{\epsilon}\right)(\mathcal{M}_n^r) \cap \mathfrak{B}_\psi^I\left(\frac{r}{2}, \dot{\epsilon}\right)(\mathcal{M}_n^r)$.
The sets follow

$$\Delta = \left\{ \begin{array}{c} \mathcal{J}\left(\mathcal{M}_n^r(q) - \mathcal{M}_n^r(\varphi), \frac{r}{2}\right) > 1 - \dot{\epsilon}_1 \text{ or} \\ \mathcal{G}\left(\mathcal{M}_n^r(q) - \mathcal{M}_n^r(\varphi), \frac{r}{2}\right) < \dot{\epsilon}_1 \text{ and} \\ \mathcal{H}\left(\mathcal{M}_n^r(q) - \mathcal{M}_n^r(\varphi), \frac{r}{2}\right) < \dot{\epsilon}_1 \end{array} \right\} \in \mathfrak{F}(I),$$

$$\nabla = \left\{ \begin{array}{c} \mathcal{J}\left(\mathcal{M}_n^r(\psi) - \mathcal{M}_n^r(\varphi), \frac{r}{2}\right) > 1 - \dot{\epsilon} \text{ or} \\ \mathcal{G}\left(\mathcal{M}_n^r(\psi) - \mathcal{M}_n^r(\varphi), \frac{r}{2}\right) < \dot{\epsilon}_1 \text{ and} \\ \mathcal{H}\left(\mathcal{M}_n^r(\psi) - \mathcal{M}_n^r(\varphi), \frac{r}{2}\right) < \dot{\epsilon}_1 \end{array} \right\} \in \mathfrak{F}(I).$$

Consider $n \in \Delta \cap \nabla$. Then

$$\begin{aligned} &\mathcal{J}(\mathcal{M}_n^r(\psi) - \mathcal{M}_n^r(q), r) \\ &\geq \mathcal{J}\left(\mathcal{M}_n^r(\psi) - \mathcal{M}_n^r(\varphi), \frac{r}{2}\right) * \mathcal{J}\left(\mathcal{M}_n^r(q) - \mathcal{M}_n^r(\varphi), \frac{r}{2}\right) \end{aligned}$$

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$\Diamond\mathscr{H}\left(\mathcal{M}_n^r(\psi_j)-\mathcal{M}_n^r(\psi),\frac{r}{3}\right)$
$<\grave{\epsilon}\Diamond\grave{\epsilon}\Diamond\grave{\epsilon}=\grave{\epsilon}.$

Take $\grave{\epsilon}=\frac{1}{n}$. Then

$$\lim_{n\to\infty}\mathscr{J}\left(\mathcal{M}_n^r(\psi^{j_p})-\mathcal{M}_n^r(\psi),r\right)=\lim_{n\to\infty}1-\frac{1}{n}=1,$$
$$\lim_{n\to\infty}\mathscr{G}(\mathcal{M}_n^r(\psi^{j_p})-\mathcal{M}_n^r(\psi),r)=\lim_{n\to\infty}\frac{1}{n}=0 \text{ and}$$
$$\lim_{n\to\infty}\mathscr{H}(\mathcal{M}_n^r(\psi^{j_p})-\mathcal{M}_n^r(\psi),r)=\lim_{n\to\infty}\frac{1}{n}=0.$$

Thus by theorem (4.3), $\psi^{j_p}\to\psi$, as $p\to\infty$.
In contrast, imagine $(\psi^{j_p})$ be the subsequence of a sequence $(\psi_j)$ in $\mathfrak{S}$ like that $(\psi^{j_p})\to\psi$ in $\mathfrak{S}$.
Let $\mathfrak{S}$ not be a compact subset of $c^I_{(\mathscr{J},\mathscr{G},\mathscr{H})}(\mathcal{M}^r)$. On the other

Thus, $\overline{\mathfrak{B}^I_\psi\left(\frac{r}{2},\grave{\epsilon}\right)(\mathcal{M}_n^r)}\subseteq\mathfrak{B}^I_\psi\left(\frac{r}{2},\grave{\epsilon}'\right)(\mathcal{M}_n^r)$.

**Theorem 4.9** *Let $\psi=(\psi_k)\in\xi$. If a sequence exists, $\psi'=(\psi_k')\in c^I_{(\mathscr{J},\mathscr{G},\mathscr{H})}(\mathcal{M}^r)$ such that $\mathcal{M}_n^r(\psi)=\mathcal{M}_n^r(\psi')$ for every $n$ relative to $I$, then $\psi\in c^I_{(\mathscr{J},\mathscr{G},\mathscr{H})}(\mathcal{M}^r)$.*

**Proof:**
Suppose $\mathcal{M}_n^r(\psi)=\mathcal{M}_n^r(\psi')$ for every $n$ relative to $I$.
Following that $\{n\in\mathbb{N}:\mathcal{M}_n^r(\psi)\neq\mathcal{M}_n^r(\psi')\}\in I$.
This implies $\{n\in\mathbb{N}:\mathcal{M}_n^r(\psi)=\mathcal{M}_n^r(\psi')\}\in\mathfrak{F}(I)$.
Whereas, because $n\in\mathfrak{F}(I)$ for every $\dot{\varpi}>0$,
$\mathscr{J}\left(\mathcal{M}_n^r(\psi)-\mathcal{M}_n^r(\psi'),\frac{\dot{\varpi}}{2}\right)=1$,
$\mathscr{G}\left(\mathcal{M}_n^r(\psi)-\mathcal{M}_n^r(\psi'),\frac{\dot{\varpi}}{2}\right)=0$ and
$\mathscr{H}\left(\mathcal{M}_n^r(\psi)-\mathcal{M}_n^r(\psi'),\frac{\dot{\varpi}}{2}\right)=0$.
Since $(\psi_k')\in c^I_{(\mathscr{J},\mathscr{G},\mathscr{H})}(\mathcal{M}^r)$,
let $I_{(\mathscr{J},\mathscr{G},\mathscr{H})}(\mathcal{M}^r)-\lim(\psi_k')=l$.
Then, each $\grave{\epsilon}\in(0,1)$ and $\dot{\varpi}>0$,

$$\Delta=\left\{\begin{array}{c}\mathscr{J}\left(\mathcal{M}_n^r(\psi')-l,\frac{\dot{\varpi}}{2}\right)>1-\grave{\epsilon}\text{ or}\\ \mathscr{G}\left(\mathcal{M}_n^r(\psi')-l,\frac{\dot{\varpi}}{2}\right)<\grave{\epsilon}\text{ and}\\ \mathscr{H}\left(\mathcal{M}_n^r(\psi')-l,\frac{\dot{\varpi}}{2}\right)<\grave{\epsilon}\end{array}\right\}\in\mathfrak{F}(I).$$

Think about the set

$$\nabla=\left\{\begin{array}{c}\mathscr{J}\left(\mathcal{M}_n^r(\psi)-l,\dot{\varpi}\right)>1-\grave{\epsilon}\text{ or}\\ \mathscr{G}\left(\mathcal{M}_n^r(\psi)-l,\dot{\varpi}\right)<\grave{\epsilon}\text{ and}\\ \mathscr{H}\left(\mathcal{M}_n^r(\psi)-l,\dot{\varpi}\right)<\grave{\epsilon}\end{array}\right\}.$$

We show that $\Delta\subset\nabla$. So for $n\in\Delta$, we have

$$\begin{aligned}&\mathscr{J}(\mathcal{M}_n^r(\psi)-l,\dot{\varpi})\\&\geq\mathscr{J}\left(\mathcal{M}_n^r(\psi)-\mathcal{M}_n^r(\psi'),\frac{\dot{\varpi}}{2}\right)*\mathscr{J}\left(\mathcal{M}_n^r(\psi')-l,\frac{\dot{\varpi}}{2}\right)\\&>1*(1-\grave{\epsilon})=1-\grave{\epsilon},\\&\mathscr{G}(\mathcal{M}_n^r(\psi)-l,\dot{\varpi})\\&\leq\mathscr{G}\left(\mathcal{M}_n^r(\psi)-\mathcal{M}_n^r(\psi'),\frac{\dot{\varpi}}{2}\right)\odot\mathscr{G}\left(\mathcal{M}_n^r(\psi')-l,\frac{\dot{\varpi}}{2}\right)\\&<0\odot\grave{\epsilon}=\grave{\epsilon}\text{ and}\\&\mathscr{H}(\mathcal{M}_n^r(\psi)-l,\dot{\varpi})\end{aligned}$$

$$\begin{aligned}&>(1-\grave{\epsilon})*(1-\grave{\epsilon})\geq(1-\grave{\epsilon}'),\\&\mathscr{G}(\mathcal{M}_n^r(\psi)-\mathcal{M}_n^r(q),r)\\&\leq\mathscr{G}\left(\mathcal{M}_n^r(\psi)-\mathcal{M}_n^r(\varphi),\frac{r}{2}\right)\odot\mathscr{G}\left(\mathcal{M}_n^r(q)-\mathcal{M}_n^r(\varphi),\frac{r}{2}\right)\\&<\grave{\epsilon}\odot\grave{\epsilon}\leq\grave{\epsilon}'\text{ and}\\&\mathscr{H}(\mathcal{M}_n^r(\psi)-\mathcal{M}_n^r(q),r)\\&\leq\mathscr{H}\left(\mathcal{M}_n^r(\psi)-\mathcal{M}_n^r(\varphi),\frac{r}{2}\right)\Diamond\mathscr{H}\left(\mathcal{M}_n^r(q)-\mathcal{M}_n^r(\varphi),\frac{r}{2}\right)\\&<\grave{\epsilon}\Diamond\grave{\epsilon}\leq\grave{\epsilon}'\end{aligned}$$

Therefore, the set

$$\left\{\begin{array}{c}\mathscr{J}(\mathcal{M}_n^r(\psi)-\mathcal{M}_n^r(q),r)>1-\grave{\epsilon}'\text{ or}\\ \mathscr{G}(\mathcal{M}_n^r(\psi)-\mathcal{M}_n^r(q),r)<\grave{\epsilon}'\text{ and}\\ \mathscr{H}(\mathcal{M}_n^r(\psi)-\mathcal{M}_n^r(q),r)<\grave{\epsilon}'\end{array}\right\}\in\mathfrak{F}(I)$$
$$\Rightarrow q=(q_k)\in\mathfrak{B}^I_\psi(r,\grave{\epsilon}')(\mathcal{M}_n^r).$$

Since $\Upsilon^j\to\psi$ as $j\to\infty$, by Theorem (4.3),
$\mathscr{J}(\mathcal{M}_n^r(\Upsilon^j)-\mathcal{M}_n^r(\Upsilon),r)\to1$, $\mathscr{G}(\mathcal{M}_n^r(\Upsilon^j)-\mathcal{M}_n^r(\Upsilon),r)\to0$ and $\mathscr{H}(\mathcal{M}_n^r(\Upsilon^j)-\mathcal{M}_n^r(\Upsilon),r)\to0$
for every $\dot{\varpi}>0$ as $n\to\infty$ .
Thus, $n\in\Delta$,

$$\begin{aligned}&\mathscr{J}(\mathcal{M}_n^r(\psi)-\mathcal{M}_n^r(\Upsilon),\dot{\varpi}+r)\\&\geq\lim_{n\to\infty}\mathscr{J}(\mathcal{M}_n^r(\Upsilon^j)-\mathcal{M}_n^r(\Upsilon),\dot{\varpi})*\mathscr{J}(\mathcal{M}_n^r(\Upsilon^j)-\mathcal{M}_n^r(\psi),r)\\&\geq1*(1-\grave{\epsilon})=1-\grave{\epsilon},\\&\mathscr{G}(\mathcal{M}_n^r(\psi)-\mathcal{M}_n^r(\Upsilon),\dot{\varpi}+r)\\&\leq\lim_{n\to\infty}\mathscr{G}(\mathcal{M}_n^r(\Upsilon^j)-\mathcal{M}_n^r(\Upsilon),\dot{\varpi})\odot\mathscr{G}(\mathcal{M}_n^r(\Upsilon^j)-\mathcal{M}_n^r(\psi),r)\\&\leq0\odot\grave{\epsilon}=\grave{\epsilon}\text{ and}\\&\mathscr{H}(\mathcal{M}_n^r(\psi)-\mathcal{M}_n^r(\Upsilon),\dot{\varpi}+r)\\&\leq\lim_{n\to\infty}\mathscr{H}(\mathcal{M}_n^r(\Upsilon^j)-\mathcal{M}_n^r(\Upsilon),\dot{\varpi})\Diamond\mathscr{H}(\mathcal{M}_n^r(\Upsilon^j)-\mathcal{M}_n^r(\psi),r)\\&\leq0\Diamond\grave{\epsilon}=\grave{\epsilon}.\end{aligned}$$

A particular $k\in\mathbb{N}$, take $\dot{\varpi}=\frac{1}{k}$. Then

$$\begin{aligned}&\mathscr{J}(\mathcal{M}_n^r(\psi)-\mathcal{M}_n^r(\Upsilon),r)\\&=\lim_{k\to\infty}\mathscr{J}\left(\mathcal{M}_n^r(\psi)-\mathcal{M}_n^r(\Upsilon),r+\frac{1}{k}\right)\geq1-\grave{\epsilon},\\&\mathscr{G}(\mathcal{M}_n^r(\psi)-\mathcal{M}_n^r(\Upsilon),r)\\&=\lim_{k\to\infty}\mathscr{G}\left(\mathcal{M}_n^r(\psi)-\mathcal{M}_n^r(\Upsilon),r+\frac{1}{k}\right)\leq\grave{\epsilon}\text{ and}\\&\mathscr{H}(\mathcal{M}_n^r(\psi)-\mathcal{M}_n^r(\Upsilon),r)\\&=\lim_{k\to\infty}\mathscr{H}\left(\mathcal{M}_n^r(\psi)-\mathcal{M}_n^r(\Upsilon),r+\frac{1}{k}\right)\leq\grave{\epsilon}\end{aligned}$$

$$\Rightarrow\left\{\begin{array}{c}\mathscr{J}(\mathcal{M}_n^r(\psi)-\mathcal{M}_n^r(\Upsilon),r)\geq1-\grave{\epsilon}\text{ or}\\ \mathscr{G}(\mathcal{M}_n^r(\psi)-\mathcal{M}_n^r(\Upsilon),r)\leq\grave{\epsilon}\text{ and}\\ \mathscr{H}(\mathcal{M}_n^r(\psi)-\mathcal{M}_n^r(\Upsilon),r)\leq\grave{\epsilon}\end{array}\right\}\in\mathfrak{F}(I)$$
$\Rightarrow\Upsilon\in\mathfrak{B}^I_\psi[r,\grave{\epsilon}](\mathcal{M}^r)$. Therefore, $\mathfrak{B}^I_\psi[r,\grave{\epsilon}](\mathcal{M}^r)$ is a closed set.

$$\leq \mathscr{H}\left(\mathcal{M}_n^r(\psi) - \mathcal{M}_n^r(\psi'), \frac{\dot{\varpi}}{2}\right) \Diamond \mathscr{H}\left(\mathcal{M}_n^r(\psi') - l, \frac{\dot{\varpi}}{2}\right)$$
$$< 0 \Diamond \grave{\epsilon} = \grave{\epsilon}$$

This suggests that $n \in \nabla$ and thus, $\Delta \subset \nabla$.
Hence $\nabla \in \mathfrak{F}(I)$ because $\Delta \in \mathfrak{F}(I)$.
Thus, $\psi = (\psi_k) \in c^I_{(\mathscr{J},\mathscr{G},\mathscr{H})}(\mathcal{M}^r)$.

**Theorem 4.10** *The closed ball $\mathfrak{B}^I_\psi[r, \grave{\epsilon}](\mathcal{M}^r)$ is a closed set in $c^I_{(\mathscr{J},\mathscr{G},\mathscr{H})}(\mathcal{M}^r)$.*

**Proof:**
Let $\Upsilon = (\Upsilon_k) \in \xi$ be like that $\Upsilon \in \mathfrak{B}^I_\psi[r, \grave{\epsilon}](\mathcal{M}^r)$.
Consequently, a sequence exists $(\Upsilon^j) = (\Upsilon^j_k) \in \mathfrak{B}^I_\psi[r, \grave{\epsilon}](\mathcal{M}^r)$ like that $\Upsilon^j$ converges to $\Upsilon$ when $j \to \infty$. Thus

$$\Delta = \left\{ \begin{array}{c} \mathscr{J}(\mathcal{M}_n^r(\Upsilon^j) - \mathcal{M}_n^r(\psi), r) \geq 1 - \grave{\epsilon} \text{ or} \\ \mathscr{G}(\mathcal{M}_n^r(\Upsilon^j) - \mathcal{M}_n^r(\psi), r) \leq \grave{\epsilon} \text{ and} \\ \mathscr{H}(\mathcal{M}_n^r(\Upsilon^j) - \mathcal{M}_n^r(\psi), r) \leq \grave{\epsilon} \end{array} \right\}.$$

# 5 Conclusions

The article examines the convergence of the sequences created by running the regular Jordan totient operator through a set of finite subsets of $\mathbb{N}$ in the setting of $\mathcal{NN}$. In order to reach a finite limit, it then applies the idea of a regular matrix to an initially non-convergent sequence. We design unique sequence spaces $c^I_{0(\mathscr{J},\mathscr{G},\mathscr{H})}(\mathcal{M}^r)$, $c^I_{(\mathscr{J},\mathscr{G},\mathscr{H})}(\mathcal{M}^r)$, $\ell^I_{\infty(\mathscr{J},\mathscr{G},\mathscr{H})}(\mathcal{M}^r)$ and $\ell_{\infty(\mathscr{J},\mathscr{G},\mathscr{H})}(\mathcal{M}^r)$ and research their connections. Future research objectives might include the creation and study of function spaces employing a generalized infinite operator.

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**Theorem 4.11** *Consider the compact subset $\mathfrak{S}$ of $c^I_{(\mathscr{J},\mathscr{G},\mathscr{H})}(\mathcal{M}^r)$ like that $\psi = (\psi_k) \notin \mathfrak{S}$. Next, there are two open sets $\mathfrak{J}, \mathfrak{K}$ in $c^I_{(\mathscr{J},\mathscr{G},\mathscr{H})}(\mathcal{M}^r)$ like that $\mathfrak{S} \subseteq \mathfrak{K}, \psi \in \mathfrak{J}$ and $\mathfrak{J} \cap \mathfrak{K} = \emptyset$.*

**Proof:**
Consider a compact subset $\mathfrak{S}$ of $c^I_{(\mathscr{J},\mathscr{G},\mathscr{H})}(\mathcal{M}^r)$ and $\psi \notin \mathfrak{S}$.
Next, for some one $s \in \mathfrak{S}$ we have $\psi \neq s$.
$c^I_{(\mathscr{J},\mathscr{G},\mathscr{H})}(\mathcal{M}^r)$ being a Hausdorff space, then for any one positive $r$ and $\grave{\epsilon} \in (0,1)$ two open balls are available $\mathfrak{J} = \mathfrak{B}^I_\psi(r, \grave{\epsilon})(\mathcal{M}^r)$ and $\mathfrak{K} = \mathfrak{B}^I_s(r, \grave{\epsilon})(\mathcal{M}^r)$ like that $\mathfrak{J} \cap \mathfrak{K} = \mathscr{J}$.
Think of the open cover $\mathfrak{K}_s = \left\{\mathfrak{B}^I_s(r, \grave{\epsilon})(\mathcal{M}^r) : s \in \mathfrak{S}\right\}$ of $\mathfrak{S}$ and $\mathfrak{S}$ is compact,
consequently, a non infinite subcover exists
$\mathfrak{S}_{s_i} = \left\{\mathfrak{B}^I_{s_i}(r, \grave{\epsilon})(\mathcal{M}^r) : s_i \in \mathfrak{S} \text{ and } i = 1, 2, 3, \ldots, j\right\}$ like that $\mathfrak{S} \subseteq \bigcup_{i=1}^j \mathfrak{K}_{s_i}$.
Taking $\mathfrak{K} = \bigcap_{i=1}^j \mathfrak{K}_{s_i}$ we have $\psi \notin \mathfrak{S}$.
Hence, $\mathfrak{J}, \mathfrak{K}$ are open sets such that $\mathfrak{S} \subseteq \mathfrak{K}$ and $\mathfrak{J} \cap \mathfrak{K} = \mathscr{J}$.

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# Properties and Applications of Klongdee Distribution in Actuarial Science

**Adisak Moumeesri1, Weenakorn Ieosanurak2,**
1Department of Statistics, Faculty of Science, Silpakorn University, Thailand
2Department of Mathematics, Faculty of Science, Khon Kaen University, Thailand

## A B S T R A C T

We have introduced a novel continuous distribution known as the Klongdee distribution, which is a combination of the exponential distribution with parameter (θ/α) and the gamma distribution with parameters (2, θ/α). We thoroughly examined various statistical properties that provide insights into probability distributions. These properties encompass measures such as the cumulative distribution function, moments about the origin, and the moment-generating function. Additionally, we explored other important measures including skewness, kurtosis, C.V., and reliability measures. Furthermore, we explore parameter estimation using nonlinear least squares methods. The numerical results presented compare the unweighted and weighted least squares (UWLS and WLS) methods, maximum likelihood estimation (MLE), and method of moments (MOM). Based on our findings, the MLE demonstrates superior performance compared to other parameter estimation methods. Moreover, we demonstrate the application of this distribution within an actuarial context, specifically in the analysis of collective risk models using a mixed Poisson framework. By incorporating the proposed distribution into the mixed Poisson model and analyzing a real-life dataset, it has been determined that the Poisson-Klongdee model outperforms alternative models in terms of performance. Highlighting its capability to mitigate the problem of overcharges, the Poisson-Klongdee model has been proven to be a valuable tool.

**Keywords** Exponential Distribution, Gamma Distribution, Parameter Estimations, Bonus-malus System, Actuarial Science

## 1. Introduction

Amixing distribution in probability theory and statistics refers to a probability distribution that results from the combination of two or more component distributions. The key concept behind a mixing distribution is that the observed random variable is generated by mixing these component distributions, where each component is assigned a specific weight or mixing proportion.

Mixture distributions find utility in various domains like finance, economics, biology, and signal processing. They provide a versatile approach to modeling intricate data that cannot be suitably characterized by a single distribution. By blending various distributions together, mixture distributions can effectively capture a broad spectrum of data patterns, including multiple modes, heavy tails, and asymmetry. This adaptability renders them a valuable instrument for faithfully depicting and studying real-world data.

Estimating and analyzing mixing distributions present intriguing challenges as it involves estimating both the mixing proportions and the parameters of the component distributions. To tackle these challenges, various statistical methods and techniques have been developed. These include maximum likelihood estimation, Bayesian inference, and expectation-maximization algorithms. These methods provide valuable tools for accurately estimating the parameters and mixing proportions of the component distributions in a mixing distribution. They enable researchers and analysts to perform robust analyses and make reliable inferences based on the observed data.

Let X be a continuous random variable that follows an exponential distribution with parameter $\lambda > 0$, denoted as $X \sim \mathrm{Exp}(\lambda)$. Its probability density function (PDF) is given by:

$$f(x;\lambda) = \lambda e^{-\lambda x}, \quad x \geq 0. \qquad (1)$$

The cumulative distribution function (CDF) of the distribution has been derived as follows:

$$F(x) = 1 - e^{-\lambda x}, \quad x \geq 0. \qquad (2)$$

Let X be a continuous random variable that follows the gamma distribution with two positive parameters, α and λ, denoted as $X \sim \mathrm{Gamma}(\alpha,\lambda)$. The probability density function (PDF) for this distribution can be written as:

$$f(x;\alpha,\lambda) = \frac{\lambda^{\alpha} x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)}, \quad x \geq 0. \qquad (3)$$

Where, Γ(α) denotes the gamma function. Notably, when $\alpha = 1$, the gamma distribution simplifies to the exponential distribution with parameter λ, represented as Exp(λ).

The CDF of a continuous random variable X following a gamma distribution with parameters $\alpha > 0$ and $\lambda > 0$ is given by:

$$F(x) = \frac{\gamma(\alpha, \lambda x)}{\Gamma(\alpha)}, \quad x > 0. \qquad (4)$$

Here, γ(α,λx) denotes the lower incomplete gamma function, and Γ(α) represents the gamma function. The two-parameter Lindley distribution combines characteristics from both the exponential and gamma distributions [1]- [8].

Ekhosuehi, Nzei, and Opone [3] proposed a Lindley distribution with two parameters by modifying the blending ratio between the exponential and gamma distributions, expressed as f(x) = wf1(x) + (1 − w)f2(x), where w = 1 1+β, and f1 and f2 represent the probability density functions of the exponential and gamma distributions, respectively. In other words,

$$f(x,\beta,\theta) = \frac{1}{1+\beta}\theta e^{-\theta x} + \frac{\beta}{1+\beta}\frac{\theta \cdot (\theta x)^{\beta-1}}{\Gamma(\beta)} e^{-\theta x}.$$

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If we have a continuous random variable $X$ that obeys the Janardan distribution (JD) [4] with two positive parameters, $\theta$ and $\alpha$, we write it as $X \sim \text{JD}(\alpha, \theta)$. The probability density function (PDF) for this distribution is as follows:

$$f(x;\alpha,\theta) = \frac{\theta^2}{\alpha(\theta+\alpha^2)}(1+\alpha x)^{-\frac{\theta}{\alpha}x}, \quad x > 0. \tag{5}$$

Its CDF is expressed as

$$F(x) = 1 - \frac{\alpha(\theta+\alpha^2)+\theta\alpha^2 x}{\alpha(\theta+\alpha^2)} exp(-\frac{\theta}{\alpha}x), \quad x > 0. \tag{6}$$

The Janardan distribution's probability density function (PDF) can be expressed as a mixture of two familiar distributions: Exp($\frac{\theta}{\alpha}$) and Gamma(2, $\frac{\theta}{\alpha}$). The following demonstrates this,

$$f(x,\alpha,\theta) = pf_1(x) + (1-p)f_2(x), \tag{7}$$

where $p = \dfrac{\theta^2}{\alpha(\theta+\alpha^2)}$, $f_1(x) = \dfrac{\theta}{\alpha}e^{-\frac{\theta}{\alpha}x}$, and $f_2(x) = \dfrac{\theta^2}{\alpha^2}xe^{-\frac{\theta}{\alpha}x}$.

Gaining a comprehensive understanding of mixing distributions and effectively utilizing them can yield valuable insights mixture of the exponential distribution ($\frac{\theta}{\alpha}$) and the gamma distribution (2, $\frac{\theta}{\alpha}$) as follows:

$$f(x;\alpha,\theta) = (1-p)f_1(x) + pf_2(x), \tag{9}$$

where $f_1(x) = \dfrac{\theta}{\alpha}e^{-\frac{\theta}{\alpha}x}$, $f_2(x) = \left(\dfrac{\theta}{\alpha}\right)^2 xe^{-\frac{\theta}{\alpha}x}$, and $p = \dfrac{\theta}{\theta+\alpha^2}$.

**Definition 1** *Define a continuous random variable $X$ to have a Klongdee distribution with two parameters $\alpha$ and $\theta$, denoted as $X \sim KD(\alpha,\theta)$, if its probability density function (PDF) is expressed as follows:*

$$f(x;\alpha,\theta) = \frac{\theta}{\theta+\alpha^2}\left(\alpha + (\frac{\theta}{\alpha})^2 x\right)e^{-\frac{\theta}{\alpha}x}, \tag{10}$$

*for all $x > 0, \alpha > 0, \theta > 0$.*

Figure 1 presents density plots depicting the Klongdee distribution for specific values of $\alpha$ and $\theta$. The observations from Figure 1 clearly indicate that the Klongdee distribution exhibits a notable characteristic of having a light tail.

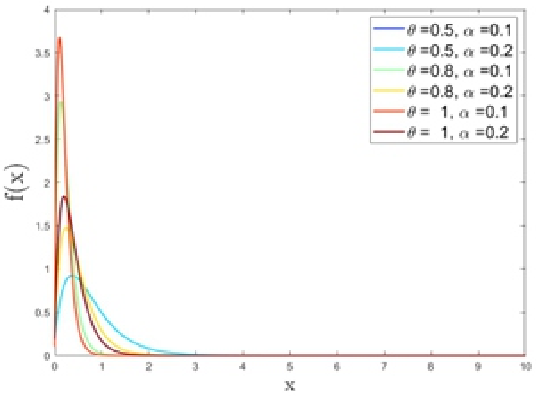


**Figure 1.** Graphs showing the PDF of the Klongdee distribution with different parameter values.

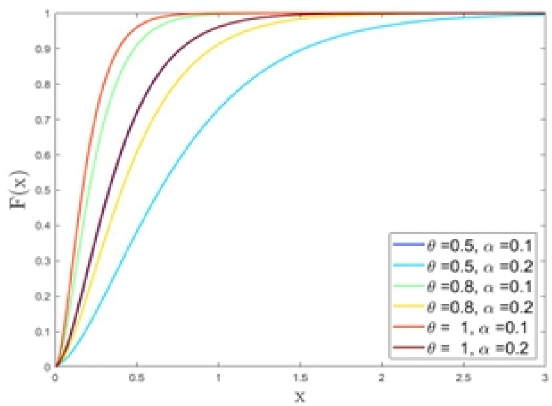


**Figure 2.** Graph showing the CDF of the Klongdee distribution with different parameter values.

To obtain the first derivative of Equation (10) with respect to $x$, we differentiate the equation accordingly:

$$\frac{d}{dx}f(x;\alpha,\theta) = \frac{\theta^2}{\alpha^3(\theta+\alpha^2)}(\alpha\theta - \alpha^3 - \theta^2 x)e^{-\frac{\theta}{\alpha}x}. \tag{11}$$

Now, based on Equation (11), we obtain

1. By setting $f'(x) = 0$, we can find the critical point of the function. Solving for $x$, we have: $x = \dfrac{\alpha\theta - \alpha^3}{\theta^2}$. For the case where $\alpha < \theta$, the value $x_0 = \frac{\alpha\theta-\alpha^3}{\theta^2}$ represents the unique critical point where $f(x)$ attains its maximum.
2. For the case when $\alpha > \theta$, we observe that $f'(x) \leq 0$, indicating that $f(x)$ is a decreasing function with respect to $x$. Consequently, the mode of the distribution described by Equation (10) is given by the value of $x$ that maximizes the PDF, which occurs at the lower bound of the support.

$$\text{Mode} = \begin{cases} \dfrac{\alpha\theta - \alpha^3}{\theta^2} & \text{if } \alpha < \theta, \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

We proceed to derive the CDF of the Klongdee distribution as presented in Theorem 2.1.

moments for random variables), the coefficient of variation (a measure of relative variability, C.V.), the coefficient of skewness (measure of distribution asymmetry), and the coefficient of kurtosis (measure of tail shape relative to normal distribution), and Reliability Measures (assessment of system/process failure probability over time).

**Theorem 2.2** *The $r^{th}$ moment about the origin of the Klongdee distribution is defined as follows:*

$$\mu_r' = \frac{r!\alpha^r(\alpha^2+(r+1)\theta)}{\theta^r(\alpha^2+\theta)}, r = 1,2,3,\ldots.$$

We can compute the first four moments around the origin for the Klongdee distribution in the following manner:

1. The first moment about the origin (mean): $\mu_1' = E[X] = \frac{\alpha(\alpha^2+2\theta)}{\theta(\alpha^2+\theta)}$.
2. The second moment about the origin : $\mu_2' = \frac{2\alpha^2(\alpha^2+3\theta)}{\theta^2(\alpha^2+\theta)}$.

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**Theorem 2.1** *The CDF of the Klongdee distribution is given by:*

$$F(x) = 1 - \frac{(\alpha^3 + \alpha\theta + \theta^2 x)}{\alpha(\alpha^2 + \theta)} e^{\frac{-\theta}{\alpha}x}, \tag{13}$$

$x > 0, \theta > 0, \alpha > 0.$

Figure 2 displays the CDF plots of the Klongdee distribution for specific values of $\alpha$ and $\theta$.

## 2.1 Statistical properties and tools

In this section, we explore and derive several properties of the Klongdee distribution, including the $r^{\text{th}}$ moment about the origin (statistical quantities describing distribution shape and characteristics), the moment generating function (efficient calculation of

the mean in the following way.

$$\mu_2 = \frac{\alpha^2(\alpha^4 + 2\theta^2 + 4\alpha^2\theta)}{\theta^2(\theta + \alpha^2)^2},$$

$$\mu_3 = \frac{2\alpha^3(2\theta^3 + \alpha^6 + 6\alpha^4\theta + 6\alpha^2\theta^2)}{\theta^3(\theta + \alpha^2)^3},$$

$$\mu_4 = \frac{3\alpha^4(8\theta^4 + 3\alpha^8 + 44\alpha^4\theta^2 + 24\alpha^6\theta + 32\alpha^2\theta^3)}{\theta^4(\theta + \alpha^2)^4}.$$

Specifically, the $2^{\text{nd}}$ moment about the mean corresponds to the variance, which is denoted by

$$\sigma^2 = \mu_2 = \frac{\alpha^2(\alpha^4 + 2\theta^2 + 4\alpha^2\theta)}{\theta^2(\theta + \alpha^2)^2}. \tag{14}$$

Other properties found for the Klongdee distribution are as follows:

1. The coefficient of variation (C.V):
   C.V. $= \frac{\sigma}{\mu_1'} = \frac{\sqrt{\alpha^4 + 2\theta^2 + 4\alpha^2\theta}}{\alpha^2 + 2\theta}$.
2. The coefficient of skewness ($\sqrt{\beta_1}$):
   $\sqrt{\beta_1} = \frac{\mu_3}{\mu_2^{3/2}} = \frac{2(2\theta^3 + \alpha^6 + 6\alpha^4\theta + 6\alpha^2\theta^2)}{(\alpha^4 + 2\theta^2 + 4\alpha^2\theta))^{3/2}}$.
3. The coefficient of kurtosis ($\beta_2$):
   $\beta_2 = \frac{3(8\theta^4 + 3\alpha^8 + 44\alpha^4\theta^2 + 24\alpha^6\theta + 32\alpha^2\theta^3)}{(\alpha^4 + 2\theta^2 + 4\alpha^2\theta)^2}$.

The moment generating function (MGF) of the Klongdee distribution can be derived as follows, as follows:

$$M_X(t) = E[e^{tX}] = \frac{\theta(\alpha^2\theta + \theta^2 - \alpha^3 t)}{(\theta + \alpha^2)(\theta - \alpha t)^2}, \quad \frac{\theta}{\alpha} > t.$$

## 2.2 Reliability Measures

In this section, our goal is to obtain formulas for the reliability measures of the Klongdee distribution. These measures encompass the survival function, failure rate function, and mean residual life function. They are essential for gaining insights into the behavior and characteristics of the Klongdee distribution, shedding light on its reliability and lifespan properties.

The survival function, denoted as $S(x)$, represents the proba-

3. The third moment about the origin:
   $\mu_3' = \frac{6\alpha^3(\alpha^2 + 3\theta)}{\theta^3(\alpha^2 + \theta)}$.
4. The fourth moment about the origin:
   $\mu_4' = \frac{24\alpha^4(\alpha^2 + 3\theta)}{\theta^4(\alpha^2 + \theta)}$.

By verifying that $\theta = \alpha^2$, we establish a direct relationship between the Klongdee distribution and the Janardan distribution. As a result, the moments about the origin of the Klongdee distribution simplify to the corresponding moments of the Janardan distribution.

To find moments around the mean, we can use the relationship between moments around the mean and moments around the origin. This relationship enables us to calculate the moment around

1. *Survival function:* $S(x) = \frac{(\alpha^3 + \alpha\theta + \theta^2 x)}{\alpha(\alpha^2 + \theta)} e^{\frac{-\theta}{\alpha}x}$.
2. *Failure rate function:* $h(x) = \frac{\theta\alpha(\alpha + (\frac{\theta}{\alpha})^2 x)}{\alpha^3 + \alpha\theta + \theta^2 x}$.
3. *Mean residual life function:* $m(x) = \frac{\alpha(\alpha^3 + 2\alpha\theta + \theta^2 x)}{\theta(\alpha^3 + 3\alpha\theta + \theta^2 x)}$.

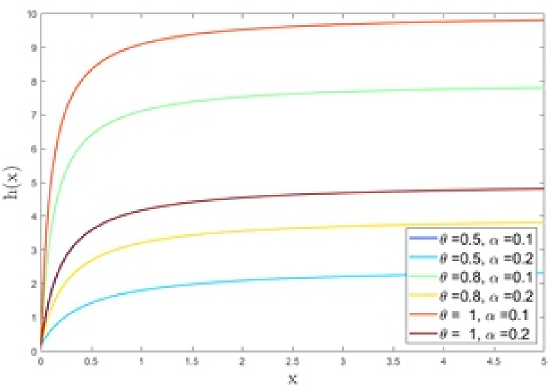


**Figure 3.** The failure rate function of the Klongdee distribution for various parameter values.

Figure 3 displays the failure rate function of the Klongdee distribution. It is worth noting some important properties of the failure rate function in relation to the Klongdee distribution:

At $x = 0$, the failure rate function takes the value $h(0) = \frac{\theta\alpha}{\theta + \alpha^2}$. Interestingly, this value is equal to the PDF evaluated at $x = 0$, denoted as $f(0)$. This indicates that the failure rate at the origin is equal to the density of the distribution at the origin.

The mean residual life function at $x = 0$, denoted as $m(0)$, corresponds to the derivative of the moment generating function evaluated at $x = 0$. In other words, $m(0) = \mu_1'$, where $\mu_1'$ represents the derivative of the first moment about the origin.

The derivative of the failure rate function, denoted as $h'(x)$, is greater than zero. This indicates that the failure rate function $h(x)$ is an increasing function of $x$, $\alpha$, and $\theta$. In other words, as $x$, $\alpha$, or $\theta$ increases, the failure rate also increases. This implies that the failure rate tends to increase with time and with higher parameter values.

bility that a Klongdee random variable exceeds a specified time $t$. It can be obtained by subtracting the CDF from 1:

$$S(x) = 1 - F(x).$$

The failure rate function, denoted as $h(t)$ or $h(x)$, provides insights into the instantaneous failure rate at time $t$. It is defined as the ratio of the PDF to the survival function: $h(x) = \dfrac{f(x)}{S(x)}$.

Lastly, the mean residual life function, denoted as $m(x)$, represents the expected remaining lifetime given that a Klongdee random variable has survived up to time $x$. It is defined as the ratio of the expected remaining lifetime to the survival function: $E[X - x|X > x]$.

**Theorem 2.3** *For $x > 0$, with parameters $\theta, \alpha > 0$, reliability measures of the Klongdee distribution are defined as follows:*

On the other hand, the mean residual life function, denoted as $m(x)$, is a decreasing function of $x$, $\alpha$, and $\theta$. This means that as $x$, $\alpha$, or $\theta$ increases, the mean residual life decreases. This implies that the expected remaining lifetime tends to decrease with time and with higher parameter values.

These properties offer valuable insights into how the failure rate function and mean residual life function behave within the Klongdee distribution.

## 3 Estimation of Parameters

Parameter estimation is a critical task in statistics and data analysis, with the primary objective of deducing the unknown parameters of a statistical model based on observed data. Its significance spans across diverse disciplines, including economics, engineering, biology, and social sciences, where comprehending the underlying parameters of a system holds vital importance for informed decision-making and drawing meaningful conclusions. The aim of parameter estimation is to determine the most optimal estimate or approximation for the unknown parameters, ensuring the best possible fit between the model and the observed data. This involves selecting an appropriate estimation method. In this section, we employ four techniques to estimate the parameters of the Klongdee distribution. These techniques include the UWLS method using the CDF, the WLS method using the CDF, the method of moments (MOM), and maximum likelihood estimation (MLE). By utilizing these techniques, we aim to obtain estimates that accurately capture the true values of the parameters. We then apply these estimation methods to the available data, ensuring that the derived estimates faithfully represent the underlying parameter values of the Klongdee distribution.

### 3.1 Unweighted least squares method via the CDF

This method is a popular technique for parameter estimation in statistical analysis. It aims to minimize the sum of squared differences between the observed CDF values and the corresponding CDF values predicted by the distribution. By focusing on the overall fit of the CDF, this method offers a straightforward and intuitive approach to estimating distribution parameters.

The UWLS method via the CDF assumes that the observed data points, denoted as $X_1, X_2, \ldots, X_n$, are generated from random variables that follow the Klongdee distribution, represented as $X_i \sim \text{KD}(\alpha, \theta)$. This assumption ensures that the data conforms to the specific Klongdee distribution with parameters $\alpha$ and $\theta$. Additionally, it is assumed that the observed data points are independent of each other, meaning that the values of $X_i$ do not depend on or influence each other within the dataset.

$$\log(F(x)) = \log\left(1 - \frac{(\alpha^3 + \alpha\theta + \theta^2 x)}{\alpha(\alpha^2 + \theta)} e^{\frac{-\theta}{\alpha} x}\right), \qquad (15)$$

$x > 0, \alpha > 0, \theta > 0$.

Consider $n$ ordered observations, where $0 < x_1 < x_2 < \ldots < x_n$. In the context of the Klongdee distribution with parameters $\alpha$ and $\theta$, these observations can be treated as independent and identically distributed random variables, denoted as $X_1, X_2, \ldots, X_n$.

where $x, \alpha, \theta > 0$.

$$F_n(x_i) = \frac{i - d}{n - 2d + 1}, \quad i = 1, 2, \ldots, n. \qquad (17)$$

For a real number $d$ such that $0 \leq d \leq 1$, we select four commonly used expressions as estimators of $F(x_i)$, where $F(x_i)$ represents the CDF at the $i$-th ordered observation $x_i$.

$$u_{ik} = \begin{cases} \frac{i}{n+1}, & k = 1 \\ \frac{i-0.3}{n+0.4}, & k = 2 \\ \frac{i-0.375}{n+0.25}, & k = 3 \\ \frac{i-0.5}{n}, & k = 4 \\ \frac{i-r}{n+r}, & k = 5. \end{cases} \qquad (18)$$

For $i = 1, 2, 3, \ldots, n$ and a real number $r$ in the range of (0, 1), our objective is to estimate the parameters $\alpha$ and $\theta$ using the UWLS method. This entails minimizing the following function:

$$E_k(\alpha, \theta) = \sum_{i=1}^{n} \left(\log(u_{ik}) - \log\left(1 - \frac{(\alpha^3 + \alpha\theta + \theta^2 x_i)}{\alpha(\alpha^2 + \theta)} e^{\frac{-\theta}{\alpha} x_i}\right)\right)^2. \qquad (19)$$

By solving the given equations for $k = 1, 2, 3, 4, 5$, we can determine the values of the unknown variables.

$$\frac{\partial}{\partial\alpha} E_k(\alpha, \theta) = 0,$$
$$\frac{\partial}{\partial\theta} E_k(\alpha, \theta) = 0.$$

Then, for $k = 1, 2, 3, 4, 5$, we can proceed with the following calculations.

$$A_k(x_i, \alpha, \theta) = \frac{(\alpha^5 - \alpha^3\theta + \alpha^2\theta^2 x_i + \theta^3 x_i) x_i e^{\frac{-\theta}{\alpha} x_i}}{(\alpha(\alpha^2 + \theta) - (\alpha^3 + \alpha\theta + \theta^2 x_i) e^{\frac{-\theta}{\alpha} x})}, \qquad (20)$$

$$\log(F(x_i)) = \log(\alpha(\alpha^2+\theta) - \log((\alpha^3+\alpha\theta+\theta^2 x)e^{\frac{-\theta x}{\alpha}})) - \log(\alpha(\alpha^2+\theta)), \quad (16)$$

$$B_k(x_i,\alpha,\theta) = \frac{(\alpha^5+\alpha^2\theta^2 x_i\theta^3 x_i))x_i e^{\frac{-v}{\alpha}x_i}}{(\alpha(\alpha^2+\theta)-(\alpha^3+\alpha\theta+\theta^2 x_i)e^{\frac{-\theta}{\alpha}x})}. \quad (21)$$

We obtain

$$\log(\hat{\alpha}) = \frac{\sum_{i=1}^{n}\left[\log\left(\hat{\alpha}(\hat{\alpha}^2+\theta)-(\hat{\alpha}^3+\hat{\alpha}\hat{\theta}+\hat{\theta}^2 x_i)e^{\frac{-\hat{\theta}}{\hat{\alpha}}x_i}-\log(u_{ik})\right)\right]A_k(x_i,\hat{\alpha},\hat{\theta})}{\sum_{i=1}^{n}A_k(x_i,\hat{\alpha},\hat{\theta})} - \log(\hat{\alpha}^2+\theta), \quad (22)$$

and

$$\log(\hat{\theta}) = -\frac{\sum_{i=1}^{n}[\log(\hat{\alpha}(\hat{\alpha}^2+\hat{\theta})-(\hat{\alpha}^3+\hat{\alpha}\hat{\theta}+\hat{\theta}2x_i)e^{\frac{-\hat{\theta}}{\hat{\alpha}}x_i})-\log(u_{ik})]B_k(x_i,\hat{\alpha},\hat{\theta})}{\sum_{i=1}^{n}B_k(x_i,\hat{\alpha},\hat{\theta})} + \log(\hat{\alpha}) + \log(\hat{\alpha}^2\theta+\hat{\theta}^2). \quad (23)$$

We utilize an iterative method to estimate $\alpha$ and $\theta$ from Equation (22) and (23), yielding the respective estimators $\hat{\alpha}$ and $\hat{\theta}$. This iterative procedure involves repeatedly updating the estimates until convergence is achieved.

### 3.2 Weighted least squares method via the CDF

The weighted least squares method via the CDF is a statistical technique used for parameter estimation. In this method, each observation in the dataset is assigned a weight, which reflects its relative importance in the estimation process. The weights are typically determined based on the characteristics of the data or the specific needs of the analysis. By incorporating these weights, the method emphasizes the observations with higher weights, giving them more influence in the estimation of the parameters. The weighted least squares method via the CDF aims to minimize the weighted sum of squared differences between the observed CDF values and the corresponding CDF values predicted by the distribution, providing a more tailored and flexible approach for parameter estimation compared to the UWLS method.

In order to address the potential issues associated with using the same weight for all data points, it is important to introduce a weighting factor to the values. This is done in Equation (19). To determine the weighting factors, a variance stabilizing transformation can be utilized. Bickel and Doksum [9] proposed an approach that relates the variance of $\log(u_{ik})$, denoted as $Var(\log(u_{ik}))$, to the uncertainty of $u_{ik}$, denoted as $Var(u_{ik})$. By considering this relationship, we obtain the following expression:

$$Var(\log(u_{ik})) = \left(\frac{\partial\log(u_{ik})}{\partial u_{ik}}\right)^2 Var(u_{ik}),$$

which gives

$$Var(\log(u_{ik})) = \left(\frac{1}{\partial u_{ik}}\right)^2 Var(u_{ik}).$$

This relationship allows us to estimate the appropriate weighting factor for each observation based on the corresponding variance. By incorporating these weighting factors into the estimation procedure, we can obtain more accurate and reliable parameter estimates.

Hence, in order to minimize the given function, we employ $w_{ik}$ as the weighting factor, defined as $w_{ik} = u_{ik}^2$.

$$E_k(\alpha,\theta) = \sum_{i=1}^{n} w_{ik}\left(\log(u_{ik}) - \log\left(1-\frac{(\alpha^3+\alpha\theta+\theta^2 x_i)}{\alpha(\alpha^2+\theta)}e^{\frac{-\theta}{\alpha}x_i}\right)\right)^2, \quad (24)$$

$k = 1, 2, 3, 4, 5$ and we solve Equation (24) in the same manner as before.
Then,

$$\log(\hat{\alpha}) = \frac{\sum_{i=1}^{n}w_{ik}\left[\log\left(\hat{\alpha}(\hat{\alpha}^2+\theta)-(\hat{\alpha}^3+\hat{\alpha}\hat{\theta}+\hat{\theta}^2 x_i)e^{\frac{-\hat{\theta}}{\hat{\alpha}}x_i}-\log(u_{ik})\right)\right]A_k(x_i,\hat{\alpha},\hat{\theta})}{\sum_{i=1}^{n}w_{ik}A_k(x_i,\hat{\alpha},\hat{\theta})} - \log(\hat{\alpha}^2+\theta), \quad (25)$$

and

$$\log(\hat{\theta}) = -\frac{\sum_{i=1}^{n}w_{ik}[\log(\hat{\alpha}(\hat{\alpha}^2+\hat{\theta})-(\hat{\alpha}^3+\hat{\alpha}\hat{\theta}+\hat{\theta}2x_i)e^{\frac{-\hat{\theta}}{\hat{\alpha}}x_i})-\log(u_{ik})]B_k(x_i,\hat{\alpha},\hat{\theta})}{\sum_{i=1}^{n}w_{ik}B_k(x_i,\hat{\alpha},\hat{\theta})} + \log(\hat{\alpha}) + \log(\hat{\alpha}^2\theta+\hat{\theta}^2). \quad (26)$$

We employ an iterative method to estimate $\alpha$ and $\theta$ from Equations (25) and (26), resulting in the estimators $\hat{\alpha}$ and $\hat{\theta}$, respectively.

By equating the theoretical first moment of the Klongdee distribution, denoted as $E[X]$, with the sample mean, denoted as $\bar{x}$, we can proceed with the estimation process.

We employ an iterative method to estimate $\alpha$ and $\theta$ from Equations (25) and (26), resulting in the estimators $\hat{\alpha}$ and $\hat{\theta}$, respectively.

### 3.3 Method of moments

The method of moments (MOM) is a widely used method for estimating the parameters of a statistical distribution based on sample moments. The main principle behind MOM is to equate the theoretical moments of the distribution with the corresponding sample moments and solve for the unknown parameters.

In the case of the Klongdee distribution, MOM involves equating the theoretical moments of the distribution (such as the mean, variance, skewness, etc.) with the sample moments calculated from the available data. By equating these moments, we can derive equations that allow us to estimate the parameters $\alpha$ and $\theta$.

In order to estimate the two parameters of the Klongdee distribution, we can utilize the first moment about the origin (mean).

Here, $\hat{\alpha}$ and $\hat{\theta}$ represent the estimators of the parameters $\alpha$ and $\theta$, respectively.

### 3.4 Maximum Likelihood Estimates

The Maximum Likelihood Estimates (MLE) method assumes that the observed data points, denoted as $x_1, x_2, \ldots, x_n$, are generated from random variables that follow the Klongdee distribution. Additionally, it is assumed that these data points are independent and identically distributed (i.i.d). The observed frequency in the sample corresponding to $X = x$ is denoted as $f_x$, where $x = 1, 2, \ldots, k$. Here, $k$ represents the largest observed value that has a non-zero frequency. It is important to note that the sum of all frequencies, $\sum_{x=1}^{k} f_x$, equals the total sample size $n$. These assumptions are crucial for applying the MLE method to estimate the parameters of the Klongdee distribution based on the observed data.

The likelihood function, denoted as $L$, of the Klongdee distribution is expressed as follows:

$$L = \left(\frac{\theta}{\theta+\alpha^2}\right)^n \prod_{x=1}^{k} \left(\alpha + (\frac{\theta}{\alpha})^2 x\right)^{f_x} e^{-\frac{\theta}{\alpha}(n\bar{x})}. \quad (27)$$

Therefore, the log-likelihood function is obtained by taking the natural logarithm of the likelihood function, resulting in:

$$\ln L = n\ln\theta - n\ln(\theta+\alpha^2) + \sum_{x=1}^{k} f_x \ln(\alpha + (\frac{\theta}{\alpha})^2 x) - \frac{\theta}{\alpha}(n\bar{x}). \quad (28)$$

By differentiating Equation (28) with respect to $\theta$ and $\alpha$, we obtain the following partial derivatives:

$$\frac{\partial \ln L}{\partial \theta} = \frac{n}{\theta} - \frac{n}{\theta+\alpha^2} + \sum_{x=1}^{k} f_x \frac{2\theta x}{\alpha^3+\theta^2 x} - \frac{n\bar{x}}{\alpha}, \quad (29)$$

$$\frac{\partial \ln L}{\partial \alpha} = \frac{-2n\alpha}{\theta+\alpha^2} + \sum_{x=1}^{k} f_x \frac{\alpha^3 - 2\theta^2 x}{\alpha^4+\alpha\theta^2 x} + \frac{\theta n\bar{x}}{\alpha^2}. \quad (30)$$

It appears that the two equations (29) and (30) cannot be directly solved. Nevertheless, the Fisher's scoring method can be employed to solve these equations, considering that we have

$$\frac{\partial^2 \ln L}{\partial \theta^2} = \frac{-n}{\theta^2} + \frac{n}{(\theta+\alpha^2)^2} + \sum_{x=1}^{k} f_x \left(\frac{2\alpha^3 x - 2\theta^2 x^2}{(\alpha^3+\theta^2 x)^2}\right), \quad (31)$$

By equating the theoretical first moment of the Klongdee distribution, denoted as $E[X]$, with the sample mean, denoted as $\bar{x}$, we can proceed with the estimation process.

$$E[X] = \frac{\alpha(\alpha^2+2\theta)}{\theta(\theta+\alpha^2)},$$
$$E[X] = \bar{X}.$$

Let us assume that $\theta = b\alpha^2$. By making this assumption, we obtain the following expression:

$$\bar{X} = \frac{\alpha(\alpha^2+2\theta)}{\theta(\theta+\alpha^2)}.$$

Therefore,

$$\hat{\alpha} = \frac{1+2b}{b(b+1)\bar{X}} \quad \text{and} \quad \hat{\theta} = \frac{1}{b}\left(\frac{1+2b}{b(b+1)\bar{X}}\right)^2,$$

## 4 Numerical results

In this section, we present the numerical results in three distinct categories. The first part highlights the application of the method to real data, demonstrating its practical utility in real-world scenarios. The second part showcases a simulated study, providing insights into the method's performance under controlled conditions. Lastly, we delve into claim modeling and insurance premium pricing, specifically examining its applicability within a bonus-malus system. By organizing the results in this manner, we provide a comprehensive overview of the method's effectiveness and its potential applications across various domains.

### 4.1 Application to real data

In this section, we proposed two real datasets: one about the waiting times of 100 bank customers [2], and the other about the survival times of 121 patients with breast cancer [10].

Table 1 presents the fittings of the Klongdee distribution, which pertain to the waiting times (in minutes) of 100 bank customers. The parameters have been estimated using the method of moments. For the purpose of comparison, the expected frequencies based on the Lindley distribution (LD) and Janardan distribution (JD) are also provided alongside those obtained from the Klongdee distribution (KD). The results highlight that the Klongdee distribution exhibits a superior fit to the data when compared to the Lindley and Janardan distributions. Moreover, Table 1 presents the expected frequencies for further analysis and comparison.

By looking at Table 1, we can see that the Kolmogorov-Smirnov (KS) statistics are very similar for all three distributions.

Also, when the chi-square ($\chi^2$) value decreases, it means that the observed and expected frequencies match better. The KD distribution has the lowest $\chi^2$ value, which shows it agrees better with the expected frequencies compared to other methods.

The survival times of 121 patients with breast cancer obtained from a large hospital in a period from 1929 to 1938 as shown in Table 2. Table 3 presents the fittings of the Klongdee distribution, which pertain to the survival times of 121 patients with breast cancer obtained from a large hospital in a period from 1929 to 1938. It's evident from the experiment that when using the WLS

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$$\frac{\partial \ln L}{\partial\theta\partial\alpha} = \frac{2n\alpha}{(\theta+\alpha^2)^2} + \sum_{x=1}^{k} f_x \frac{-6\alpha^2\theta x}{(\alpha^3+\theta^2 x)^2} + \frac{n\bar{x}}{\alpha^2}, \quad (32)$$

$$\frac{\partial^2 \ln L}{\partial\alpha^2} = \frac{2n\alpha^2 - 2n\theta}{(\theta+\alpha^2)^2} - \sum_{x=1}^{k} f_x \left(\frac{\alpha^6 - 10\alpha^3\theta^2 x - 2\theta^4 x^2}{(\alpha^4+\alpha\theta^2 x)^2}\right) - \frac{2\theta n\bar{x}}{\alpha^3}. \quad (33)$$

The equation for estimating $\hat{\theta}$ and $\hat{\alpha}$ can be solved using the following expressions:

$$\begin{bmatrix} \frac{\partial^2 lnL}{\partial\theta^2} & \frac{\partial^2 lnL}{\partial\theta\partial\alpha} \\ \frac{\partial^2 lnL}{\partial\alpha\partial\theta} & \frac{\partial^2 lnL}{\partial\alpha^2} \end{bmatrix} \begin{bmatrix} \hat{\theta} - \theta_0 \\ \hat{\alpha} - \alpha_0 \end{bmatrix} = \begin{bmatrix} \frac{\partial lnL}{\partial\theta} \\ \frac{\partial lnL}{\partial\alpha} \end{bmatrix}, \quad (34)$$

where $\theta_0$ and $\alpha_0$ are the initial values of $\theta$ and $\alpha$ respectively. This equation is solved iteratively until sufficiently accurate estimates of $\hat{\theta}$ and $\hat{\alpha}$ are obtained.

method with $k = 2$, the chi-square value is minimized. The chi-square values for the WLS and UWLS methods are quite similar. Conversely, when employing the WLS method with $k = 4$, the KS test yields the lowest value, while the KS test values remain similar for the WLS and UWLS methods. However, it's worth noting that the MOM method produces a notably higher KS test result.

#### 4.1.1 A simulated study

Based on the waiting time data presented in Table 1, we estimate the parameter values $(\theta, \alpha) = (0.0077, 0.0422)$ using the method of moments. To generate samples from the specified distribution with the parameter values $(\theta, \alpha) = (0.0077, 0.0422)$, we utilize the acceptance-rejection method [11]. In this process, we set the sample size to $1,000$ and perform $1,000,000$ iterations for each method to ensure sufficient sample generation and accurate representation of the distribution.

**Table 1.** Waiting times (in minutes) of 100 bank customers for observed and expected frequencies.

| Waiting time (minutes) | Observed frequency | Expected frequency | | |
|---|---|---|---|---|
| | | LD | JD | KD (MOM) |
| 0-5 | 30 | 30.39 | 30.16 | 29.92 |
| 5.01-10 | 32 | 30.69 | 30.92 | 29.79 |
| 10.01-15 | 19 | 19.21 | 19.32 | 19.18 |
| 15.01-20 | 10 | 10.28 | 10.28 | 10.63 |
| 20.01-25 | 5 | 5.08 | 5.05 | 5.46 |
| 25.01-30 | 1 | 2.39 | 2.37 | 2.67 |
| 30.01-35 | 2 | 1.09 | 1.07 | 1.27 |
| 35.01-40 | 1 | 0.49 | 0.47 | 0.59 |
| total | 100 | 99.62 | 99.64 | 99.51 |
| Estimated parameters | | $\hat{\theta} = 0.1897$ | $\hat{\theta} = 0.2139$ | $\hat{\theta} = 0.0077$ |
| | | | $\hat{\alpha} = 1.1189$ | $\hat{\alpha} = 0.0422$ |
| $\chi^2$ | | 2.1711 | 2.2499 | 1.9956 |
| d.f. | | 6 | 5 | 5 |
| KS test | | 0.90038 | 0.90043 | 0.90049 |

**Table 2.** Survival times of 121 patients with breast cancer.

| Survival times | Observed frequency |
|---|---|
| 0.3 -19.5125 | 32 |
| 19.5025 - 38.7250 | 26 |
| 38.7150 - 57.9375 | 28 |
| 57.9275 - 77.1500 | 13 |
| 77.1400 - 96.3625 | 8 |
| 96.3525 - 115.5750 | 6 |
| 115.5650 - 134.7875 | 6 |
| 134.7775 - 154.000 | 2 |
| total | 121 |

squared test statistic ($\chi^2$) is utilized, which is defined as follows:

$$\chi^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i},$$

where, $O_i$ represents the observed value and $E_i$ represents the expected value for each category. By calculating the Chi-squared test statistic, we can evaluate the discrepancy between the observed and expected values and determine the goodness of fit of the estimation methods. Furthermore, we employ the Kolmogorov-Smirnov test, which can be described as follows:

$$\max |F_O(x) - F_T(x)|,$$

where $F_O(x)$ denotes the CDF observed in a random sample of n observations, while $F_T(x)$ pertains to the theoretical frequency distribution.

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Suppose we have a requirement to sample a random value $x_i$ from the Klongdee distribution, denoted as $f(x)$, in order to calculate the function value for any given $x$. To accomplish this, we define an auxiliary distribution function as a uniform distribution. We select a value for the "envelope constant" ($m > 0$), which is used to scale the auxiliary distribution and create the "blanket function" denoted as $m \cdot g(x)$. The choices of $g$ and $m$ must satisfy the condition $m \cdot g(x) \geq f(x)$ for all $x$. For the waiting time data, it is possible to choose $m \geq 2.8$. In this simulation, we specifically choose the value of $m = 2.8$ to ensure the blanket function adequately covers the Klongdee distribution.

In this section, we perform a comparative analysis of four distinct parameter estimation methods: the UWLS method via the CDF, the WLS method via the CDF, the MLE and the MOM. The results of this simulation study are presented in Table 4, providing a comprehensive overview of the performance of each method. Additionally, Table 5 displays the corresponding chi-square values associated with each estimation method, further aiding in the evaluation of their effectiveness.

In our analysis, we employ the Chi-squared test as a metric to assess the performance of the estimation methods [12]. The Chi-

The analysis of Table 4 reveals that among the UWLS methods, the one with $k = 4$ stands out as the most favorable. Similarly, within the WLS methods, the one with $k = 1$ emerges as the top performer. These findings suggest that these specific parameter configurations yield the most accurate and reliable estimations within their respective methods.

We proceed to compare the optimal UWLS, the optimal WLS, MLE, and MOM methods. Using a significance level of $0.05$, we obtain $\chi^2_{0.05,5} = 11.07$ from Table 5. Upon examination of the table, it becomes apparent that the estimates obtained through UWLS ($k = 4$), WLS ($k = 1$), and MLE are all below $11.07$ and display a high degree of similarity. However, the Chi-squared value associated with the MLE method surpasses the others, indicating its superiority in terms of goodness of fit.

### 4.1.2 Bonus-Malus System

In this section, we mix the Poisson distribution with the proposed distribution and apply the mixed distribution to an actual dataset. Our goal is to demonstrate that our mixed model is superior to other competing models in terms of how well it fits the data. Additionally, we present a model for calculating automobile insurance premiums under the bonus-malus system.

**Table 3.** Survival times of 121 patients with breast cancer for observed and expected frequencies.

| Observed frequency | Expected frequency | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | UWLS | | | | | WLS | | | | | MOM | MLE |
| | k=1 | k=2 | k=3 | k=4 | k=5 | k=1 | k=2 | k=3 | k=4 | k=5 | | |
| 32 | 31.47 | 31.47 | 30.35 | 30.35 | 30.35 | 32.02 | 30.19 | 30.62 | 31.38 | 33.08 | 28.72 | 28.09 |
| 26 | 28.88 | 28.88 | 27.57 | 27.57 | 27.57 | 27.99 | 28.28 | 25.46 | 30.60 | 28.55 | 25.76 | 27.77 |
| 28 | 21.61 | 21.61 | 21.09 | 21.09 | 21.09 | 20.93 | 21.58 | 19.51 | 22.53 | 21.00 | 20.27 | 21.80 |
| 13 | 14.70 | 14.70 | 14.84 | 14.84 | 14.84 | 14.46 | 14.97 | 14.23 | 14.78 | 14.26 | 14.87 | 15.46 |
| 8 | 9.47 | 9.47 | 9.93 | 9.93 | 9.93 | 9.53 | 9.84 | 10.05 | 9.09 | 9.22 | 10.45 | 10.34 |
| 6 | 5.89 | 5.89 | 6.42 | 6.42 | 6.42 | 6.08 | 6.23 | 6.93 | 5.37 | 5.78 | 7.13 | 6.66 |
| 6 | 3.57 | 3.57 | 4.06 | 4.06 | 4.06 | 3.79 | 3.85 | 4.70 | 3.08 | 3.53 | 4.76 | 4.18 |
| 2 | 2.12 | 2.12 | 2.52 | 2.52 | 2.52 | 2.32 | 2.33 | 3.14 | 1.74 | 2.12 | 3.12 | 2.57 |
| $\sum = 121$ | | | | | | | | | | | | |
| $\hat{\theta}$ | 0.0006 | 0.0006 | 0.0006 | 0.0006 | 0.0006 | 0.0007 | 0.0006 | 0.0008 | 0.0005 | 0.0008 | 0.0006 | 0.0005 |
| $\hat{\alpha}$ | 0.0189 | 0.0189 | 0.0199 | 0.0199 | 0.0199 | 0.0228 | 0.0177 | 0.0288 | 0.0147 | 0.0237 | 0.0210 | 0.0146 |
| $\chi^2$ | 4.2852 | 4.2852 | 4.1062 | 4.1062 | 4.1062 | 4.2573 | **4.0626** | 5.1946 | 5.2456 | 4.6063 | 5.0436 | 4.3233 |
| df | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| KS test | 0.0334 | 0.0334 | 0.0578 | 0.0578 | 0.0578 | 0.0418 | 0.0492 | 0.0860 | 0.0329 | 0.0300 | 0.0930 | 0.0689 |

**Table 4.** Estimation Results for $\theta = 0.0077$ and $\alpha = 0.0422$ with a Sample Size of 1,000 and Waiting Time Data using UWLS, WLS, MLE, and MOM Methods.

| Methods | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| MLE | | MOM | | UWLS | | | WLS | |
| $\theta$ | $\alpha$ | $\theta$ | $\alpha$ | $k$ | $\theta$ | $\alpha$ | $\theta$ | $\alpha$ |
| | | | | 1 | 0.008620 | 0.042426 | 0.007211 | 0.041406 |
| | | | | 2 | 0.008110 | 0.042066 | 0.006687 | 0.042586 |
| 0.005480 | 0.029499 | 0.008684 | 0.041493 | 3 | 0.007787 | 0.042681 | 0.007731 | 0.041440 |
| | | | | 4 | 0.007353 | 0.042069 | 0.006233 | 0.042617 |
| | | | | 5 | 0.008153 | 0.043147 | 0.008287 | 0.042139 |

To enhance the modeling of the claim frequency distribution in automobile insurance, each policyholder is assigned a risk parameter that signifies their risk of experiencing an accident. This risk parameter is considered a random variable that varies among policyholders and follows a prior distribution. One proposed approach for modeling the frequency distribution involves mixing the Poisson distribution with the Klongdee distribution.

**Mixing distribution** : Assuming that the probability mass function (PMF) for the count of claims, denoted by $y$, is represented by the Poisson distribution with a parameter value of $\lambda$, the PMF can be formulated as follows:

$$f(y|\lambda) = \frac{e^{-\lambda}\lambda^y}{y!}, \quad y = 0, 1, 2, \ldots, \lambda > 0. \tag{35}$$

The expected value or mean of the Poisson random variable is $E[Y|\lambda] = \lambda$.

The average number of claims made by a policyholder reflects their underlying risk, represented as a constant value denoted by

where $\alpha > 0, \theta > 0$ and $y = 0, 1, 2, \ldots$.

**Model fitting** : [13] presented a dataset on the distribution of automobile insurance policyholders according to the number of claims. This dataset is observed to be highly skewed towards the right and over-dispersed, where the variance is greater than the mean. The Poisson distribution (PD) is widely recognized as an unsuitable option for automobile insurance claims due to its mean and variance restriction. As a result, a mixed Poisson distribution with a prior distribution is preferred in such cases. Table 6 presents a comparison between the mixed Poisson model and the proposed model, including the Poisson-Lindley distribution (PLD), using maximum likelihood parameter estimation. The Chi-square statistic values indicate that the Poisson-Klongdee distribution (PKD) offers a superior fit to the dataset compared to other distributions.

**Bonus − Malus premium** : Many countries have implemented a bonus-malus system (BMS) that rewards policyholders with no claims and punishes those with claims. Thus, the follow-

$\lambda$. In the proposed model, $\lambda$ is assumed to adhere to the Klongdee distribution with parameters $\alpha$ and $\theta$. This distribution can be characterized by the probability density function (PDF) of $\lambda$, as shown below:

$$\pi(\lambda) = \frac{\theta}{\theta+\alpha^2}\left(\alpha + (\frac{\theta}{\alpha})^2\lambda\right)e^{-\frac{\theta}{\alpha}\lambda}, \tag{36}$$

for all $\lambda > 0, \alpha > 0, \theta > 0$. Outlined below is the procedure to derive the mixed Poisson distribution in conjunction with the Klongdee distribution:

$$\begin{aligned} f(y) &= \int_0^\infty f(y|\lambda)\,\pi(\lambda)\,d\lambda \\ &= \int_0^\infty \frac{e^{-\lambda}\lambda^y}{y!}\cdot\frac{\theta}{\theta+\alpha^2}\left(\alpha+(\frac{\theta}{\alpha})^2\lambda\right)e^{-\frac{\theta}{\alpha}\lambda}\,d\lambda \\ &= \frac{\theta\alpha^y[\alpha^2(\alpha+\theta)+\theta^2(y+1)]}{(\theta+\alpha^2)(\alpha+\theta)^{y+2}}, \end{aligned} \tag{37}$$

ing year's premium is determined by the policyholder's history up to the current year, regardless of the claim size. [14] calculated the BMS premium using the following formula:

$$\text{Premium}_{t+1} = \frac{E_{\pi^*(\lambda|n)}[l(\lambda)]}{E_{\pi(\lambda)}[l(\lambda)]}\times 100, \tag{38}$$

where $l(\lambda) = \sum_{n=0}^{\infty} nP(N=n|\lambda)$, $\pi(\lambda)$ represents the prior distribution, and $\pi^*(\lambda|n)$ signifies the posterior distribution. Consequently, when working with a sequence of independent and identically distributed claims denoted as $n = (n_1, n_2, \ldots, n_t)$, it becomes clear that deriving the posterior distribution is a simple process achieved by dividing the mixing distribution by the marginal distribution, as depicted below.

$$\pi^*(\lambda|n) = \frac{P(n|\lambda)\pi(\lambda)}{\int_0^\infty P(n|\lambda)\pi(\lambda)d\lambda}. \tag{39}$$

**Table 5.** Chi-squared Test Statistic ($\chi^2$) Results for UWLS ($k = 4$), WLS ($k = 1$), MLE, and MOM Methods with Waiting Time Data.

| Waiting time (minutes) | $f$ | Methods | | | |
|---|---|---|---|---|---|
| | | UWLS ($k=4$) | WLS ($k=1$) | MLE | MOM |
| 0 - 5 | 275 | 288.7408 | 286.8614 | 288.3657 | 342.1238 |
| 5 - 10 | 303 | 291.7746 | 291.3072 | 305.4621 | 319.1345 |
| 10 - 15 | 198 | 193.2380 | 193.6271 | 196.3265 | 181.9561 |
| 15 - 20 | 124 | 110.4687 | 111.0644 | 107.4397 | 88.4414 |
| 20 - 25 | 60 | 58.5476 | 59.0564 | 54.2461 | 39.6778 |
| 25 - 30 | 22 | 29.6272 | 29.9814 | 26.0914 | 16.9610 |
| 30 - 35 | 11 | 14.5315 | 14.7525 | 12.1485 | 7.0194 |
| 35 - 40 | 7 | 6.9688 | 7.0974 | 5.5265 | 2.8384 |
| Estimated parameters | $\theta$ | 0.0074 | 0.0072 | 0.0055 | 0.0087 |
| | $\alpha$ | 0.04207 | 0.0414 | 0.0295 | 0.0415 |
| | $\chi^2$ | 5.7185 | 5.6608 | 4.9595 | 49.9615 |
| | KS test | 0.0172 | 0.0181 | 0.0158 | 0.0832 |

**Table 6.** Number of claims in automobile insurance.

| Number of claim | Observed frequency | Expected frequency | | |
|---|---|---|---|---|
| | | PD | PLD | PKD |
| 0 | 63,232 | 63,094.32 | 63,252.68 | 63,234.36 |
| 1 | 4333 | 4590.55 | 4292.03 | 4326.26 |
| 2 | 271 | 167.00 | 290.30 | 277.25 |
| 3 | 18 | 4.05 | 19.58 | 17.05 |
| 4 | 2 | 0.07 | 1.32 | 1.02 |
| 5 | 0 | 0.00 | 0.09 | 0.06 |
| total | 67,856 | 67,856 | 67,856 | 67,856 |
| Estimated parameters | | $\hat{\theta} = 0.0728$ | $\hat{\theta} = 14.6238$ | $\hat{\theta} = 643.2161$, $\hat{\alpha} = 34.7107$ |
| $\chi^2$ | | 177.9421 | 2.2507 | 1.2083 |

$$\pi^*(\lambda|n) = \frac{\alpha^2(t+\frac{\theta}{\alpha})^{n+2}}{\Gamma(n+1)\left[\alpha^3(t+\frac{\theta}{\alpha})+\theta^2(n+1)\right]}e^{-(t+\frac{\theta}{\alpha})\lambda}\left(\alpha+\frac{\theta^2}{\alpha^2}\lambda\right)\lambda^n. \tag{40}$$

$$\text{Premium}_{t+1} = \frac{\theta(n+1)(\alpha^2+\theta)}{(\alpha t+\theta)(\alpha^2+2\theta)}\cdot\frac{\alpha^2(\alpha t+\theta)+\theta^2(n+2)}{\alpha^2(\alpha t+\theta)+\theta^2(n+1)}\cdot 100. \tag{41}$$

The posterior distribution of the Poisson-Klongdee distribution can be expressed in the following form:

By utilizing Equation (41), we compute the bonus-malus premiums only considering the frequency component. The outcomes we obtain are showcased in Table 7.

According to Table 7, policyholders who do not submit any claims in the first year receive a bonus equal to 5.95% of the base premium. On the other hand, policyholders who make a single claim during the first year face a penalty of 76.19% of the base premium. Claim-free policyholders enjoy lower premiums, whereas premiums increase for policyholders who file claims.

In order to facilitate comparisons, we have computed the bonus-malus premiums using the traditional Poisson-Lindley model (see [15], for details). The results are displayed in Table 8.

Based on the results showcased in Table 8, policyholders who refrain from submitting any claims in the first year receive a bonus equivalent to 6.74% of the base premium. In contrast, policyholders who file a single claim in the first year incur a malus amounting to 85.92% of the base premium.

The Poisson-Klongdee model demonstrates a lower level of penalization in comparison to traditional Poisson-Lindley models, highlighting its ability to alleviate the issue of overcharges.

## 5 Conclusions

A two-parameter continuous distribution called the Klongdee distribution has been introduced. This distribution's properties, including its CDF, expected value, $r^{\text{th}}$ moment, and parameter estimation using nonlinear least squares methods, MLE, and MOM, have been proposed.

**Table 7.** Bonus-malus premium using Poisson-Klongdee model.

| t | Number of claims | | | | | |
|---|---|---|---|---|---|---|
| | 0 | 1 | 2 | 3 | 4 | 5 |
| 0 | 100.00 | | | | | |
| 1 | 94.05 | 176.19 | 253.60 | 328.66 | 402.37 | 475.26 |
| 2 | 88.73 | 166.77 | 240.44 | 311.90 | 382.09 | 451.48 |
| 3 | 83.94 | 158.26 | 228.54 | 296.73 | 363.71 | 429.92 |
| 4 | 79.63 | 150.55 | 217.72 | 282.92 | 346.98 | 410.29 |
| 5 | 75.71 | 143.52 | 207.84 | 270.31 | 331.68 | 392.35 |
| 6 | 72.14 | 137.09 | 198.79 | 258.74 | 317.65 | 375.87 |
| 7 | 68.88 | 131.19 | 190.47 | 248.10 | 304.72 | 360.70 |

**Table 8.** Bonus-malus premium using Poisson-Lindley model.

| t | Number of claims | | | | | |
|---|---|---|---|---|---|---|
| | 0 | 1 | 2 | 3 | 4 | 5 |
| 0 | 100.00 | | | | | |
| 1 | 93.26 | 185.92 | 278.08 | 369.81 | 461.17 | 552.22 |
| 2 | 87.37 | 174.23 | 260.67 | 346.74 | 432.50 | 517.99 |
| 3 | 82.17 | 163.92 | 245.30 | 326.37 | 407.17 | 487.72 |
| 4 | 77.56 | 154.75 | 231.63 | 308.24 | 384.61 | 460.77 |
| 5 | 73.43 | 146.55 | 219.40 | 292.01 | 364.41 | 436.63 |
| 6 | 69.72 | 139.18 | 208.39 | 277.39 | 346.21 | 414.87 |
| 7 | 66.37 | 132.50 | 198.42 | 264.16 | 329.74 | 395.17 |

The simulation results are divided into two parts. In the first part, the Klongdee distribution is applied to a data set representing waiting times in order to test its goodness of fit. The results show that the Klongdee distribution provides better fits compared to the earlier fits of the Lindley distribution and Janardan distribution. In the second part, we obtain the parameters for generating data based on the results from the first part. We then compare the performance of four methods using the Chi-squared test. Our analysis concludes that the MLE estimators outperform the UWLS, WLS, and MOM estimators in terms of performance.

In the context of actuarial science, we propose the mixed Poisson with Klongdee distribution as a model for claim modeling. We utilize this mixed distribution to develop a pricing model for insurance premiums based on the BMS. The findings indicate that the Poisson-Klongdee distribution has the ability to address the problem of overcharging.

The Klongdee distribution is specifically designed to suit right-skewed data. However, its application to datasets exhibiting different skewness characteristics can result in inadequate fit and inaccurate outcomes. It's imperative to thoroughly evaluate the skewness of the data before selecting the distribution. When dealing with data that lacks right-skewness, considering alternative distributions like the normal or gamma distribution is advisable, as it has the potential to yield more accurate results. In forthcoming research, we aim to investigate the feasibility of relaxing specific assumptions in our applications, notably the assumption of independence. This exploration will involve assessing the potential implications of such relaxations on our conclusions. Furthermore, our future endeavors will encompass expanding the model to encompass a wider array of assumptions and complexities. This extension aims to elevate the applicability and robustness of our findings to a broader range of scenarios. In conclusion, comprehending the intricacies of mixture distributions is instrumental in refining decision-making processes. By adeptly capturing complex patterns and revealing latent structures within datasets, these distributions empower us to formulate strategies that are not only well-informed but also precise in real-world applications.

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